Checking your answers in mathematics and physics

Problem statements and initial solutions

Problem A: Division

Evaluate 3141 / 16.

Use long division.

$$\begin{array}{r|l} & 196 \\ \hline 16 & 3141 \\ - & 16\downarrow\downarrow \\ \hline & 154\downarrow \\ - & 144\downarrow \\ \hline & 101 \\ - & 96 \\ \hline & 5 = R \\ \hline \end{array}$$

Thus 3141 / 16 = 196 $^5/_{16}$.

Problem B: Complex numbers

Evaluate $(-2\sqrt{3}+2i) / (-1+i\sqrt{3})$.

Multiply numerator and denominator by the complex conjugate of the denominator.

$$\frac{-2\sqrt{3}+2i}{-1+\sqrt{3}i} = \frac{-2\sqrt{3}+2i}{-1+\sqrt{3}i}\,\frac{-1-\sqrt{3}i}{-1-\sqrt{3}i} = \frac{+2\sqrt{3}+6i-2i-2\sqrt{3}i^2}{+1+\sqrt{3}i-\sqrt{3}i-3i^2}\,\frac{4\sqrt{3}+4i}{4} = \sqrt{3}+i\,.$$

Problem C: Ultrasonography

Fill in the blanks.

Material	Density, ρ (kg/m³)	Acoustic velocity, v (m/s)	Acoustic impedance, Z (kPa·s/m)
Muscle	1075		1.71
Brain	1025	1540	1.58
Bone	1910	4080	
Lens of eye		1136	1.84

Recall the applicable formula.

For muscle: $v = Z / \rho = 1.71\times10^6 / 1075$ m/s = 1590.698 m/s ≈ 1591 m/s.

For bone: $Z = \rho\, v = 1910 \times 4080 = 7792800$ kg/(m²·s) ≈ 7.79 kPa·s/m.

For the lens: $\rho = Z / v = 1.84\times10^6 / 1136$ kg/m³ = 1619.718 kg/m³ ≈ 1620 kg/m³.

Problem D: Electricity

How much power is consumed by a 10 A appliance running directly off domestic mains electricity in Australia?

Recall data and formula.

It may be recalled that domestic power in Australia is nominally 240 V. This is the nominal value of the RMS (root-mean-square) value of the AC (alternating current) signal; the true value may fluctuate slightly around this over the course of time.

The formula for power consumption is simply $P = V I$.

Hence, P = 240 V × 10 A = 2400 W.

Tactic 1: Re-work the solution

Read through worked solution

Reading through a worked solution is the **most common approach**. It can be the quickest, but there is a high risk of overlooking (not finding) errors in the solution. The chance of picking up errors is improved if it is already strongly suspected that an error has occurred (either through one of the other tactics discussed below, or if the final answer does not match options in a multiple-choice question).

Independently complete the solution again

Starting on a fresh piece of paper, work out the entire problem again from scratch. This is time-consuming — and therefore seldom implemented — but it is a very good way of being assured that no 'accidental' calculation or transcription or data entry errors are present. For example, assuring that: the correct values, as specified in the problem statement, are used; positive or negative signs are not 'dropped'; algebraic expansions are performed correctly; the right buttons are pressed on an electronic calculator; appropriate cell references are entered in spreadsheet formulæ.

The rationale for this is that such mistakes are essentially 'random', so that for a competent operator **the chance of making exactly the same mistake(s) on two independent solution attempts is small**.

This approach will not pick up systematic errors, especially conceptual mistakes. These could include confusion between the "sin" (sine) and "sinh" (hyperbolic sine) buttons on a an electronic calculator; forgetting to convert angles in degrees to radians (or *vice versa*) before computing trigonometric function; misremembering a formula or procedure. For example: in Problem A, misunderstanding the concept of a remainder to incorrectly obtain 196 $^{5}/_{3141}$; in Problem B, incorrectly thinking that multiplying i by itself simply results in 'cancellation'; in Problem C, incorrectly recalling that a 'correction' for the density of water (~1000 kg/m^3) is required, yielding an incorrect formula, $v = Z / (\rho - \rho_{water})$. [The latter should be quickly discarded after finding that it is not consistent with the data given for brain matter.]

The most difficult mistakes to pick up would be those that have little impact — or even no effect — on the final result. For Problem C, incorrectly recalling that a 'correction' for the density of air (~1 kg/m^3) is required, yielding an incorrect formula, $v = Z / (\rho - \rho_{air})$, would be less obvious, as the effect on the final answer would be small, possibly not even showing up after rounding to an appropriate number of significant figures.

Occasionally, repeating the same mistake twice (or a combination of different mistakes) will still lead to exactly the right final answer, for the given problem. Sometimes this is done 'wilfully', where a student might 'ignore' a negative sign in an intermediate result that imply a physical impossibility.

Tactic 2: Try order-of-magnitude calculations.

A **simpler 'approximate' version of the problem** can often be constructed that can be solved with greater confidence. If the answer to the original problem is close to the answer to the simple approximation of the problem, then this agreement or 'concordance' provides a measure of reassurance.

Conversely, if the values are wildly dissimilar, then it should be expected that one of the values is wrong. As the 'approximate' version is inherently easier to solve, the initial suspicion would be that the proposed answer to the original problem is erroneous.

Problem A: Division

We notice that $3141 / 16 \approx 3200 / 16$, and we recall that $32/16 = 2$.

Thus $3141 / 16 \approx 3200 / 16 = 200$.

This is close to the answer we found above for the original problem, namely 196 $^{5}/_{16}$.

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As 3200 is 'a bit' bigger (relatively) than 3141, we expect the answer to the original problem to be 'a bit' smaller (relatively) than 200.

This is a 'low resolution' test, as it would not identify a mistake of relatively small magnitude, such as neglecting to include the remainder in the final result (*i.e.* incorrectly claiming the answer to be 196 exactly). However, we would feel fairly confident in rejecting "299" or "15" as possible answers to the original problem.

So far we have an upper bound of 200 on the answer to the original problem. We could also obtain a lower bound by evaluating $3141 / 16 \approx 3141 / 20 = 314.1 / 2 = 157.05$.

If we approximate more than one value in the question (by proportionally similar amounts), then we still obtain a result that approximates the answer to the original problem, but it cannot be readily interpreted as an upper or lower bound. For example,
$3141 / 16 \approx 3000 / 15 = 200$.

Problem B: Complex numbers

All of the values in the problem statement have magnitudes of order 10^0, so we expect the answer to also involve magnitudes up to order 10^0, possibly with some smaller values (due to the possibility of cancellation of real×real with imaginary×imaginary terms). Our proposed answer fits this expectation, so we are reassured.

This would not have allowed us to identify a mistake in the final division-by-4 operation, incorrectly yielding a result of $\sqrt{3}+4i$. However, we would be able to recognise $^{413}/_4 + i$ (from misreading the square-root sign) as incorrect.

Problem C: Ultrasonography

Here there is an extra complication because of the units conversion. Provided we express Z in base SI units [kg/(m²·s) = Pa·s/m], then we have the following.

For muscle: $v = Z / \rho \sim 10^6 / 10^3$ m/s $= 10^3$ m/s.

For bone: $Z = \rho\, v \sim 10^3 \times 10^3 = 10^6$ kg/(m²·s) $= 10^0$ kPa·s/m.

For the lens: $\rho = Z / v \sim 10^6 / 10^3$ kg/m³ $= 10^3$ kg/m³.

Problem D: Electricity

The multiplication is fairly trivial in this problem, but the general principle can still be applied:

$P \sim 10^2$ V $\times 10^1$ A $= 10^3$ W.

Tactic 3: Check dimensions.

Dimensions must be consistent. Operations adding or subtracting or equating quantities must have the same units. Strictly speaking, many common mathematical functions only operate on dimensionless numbers (*e.g.* logarithms, exponentials, inverse-sine). Sometimes this seems to be 'ignored', as when calculating sound levels in decibels or computing pH from nominal hydrogen ion concentration, but this is because strictly-incorrect shorthand notations are being used in which unit conversion factors are implied (without being actually written in).

Problem A: Division

The values in the problem statement are dimensionless, so the answer must also be dimensionless.

Problem B: Complex numbers

The values in the problem statement are dimensionless, so the answer must also be dimensionless.

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Problem C: Ultrasonography

This problem is somewhat unusual in that the required dimensions of the answers are explicitly indicated by the column headings in the table.

In fact, this fortuitously permits us to make an educated guess at the relationship between the variables, in the event that we could not remember (or were not confident about) the true formula.

Recall that Pa is a unit of pressure, which is defined as a force per unit area, thus Pa = N/m^2. Also, the product of mass by acceleration yields a force ($F = m\,a$), reminding us that N = $kg \cdot m/s^2$. Hence $Pa \cdot s/m = [(kg \cdot m/s^2)/(m^2)] \cdot s/m = kg/(m^2 \cdot s)$.

Given velocity in m/s, and density in kg/m^3, it is natural to expect that Z might be obtained from their product, as the units would then be consistent, namely:
$(kg/m^3) \cdot (m/s) = kg/(m^2 \cdot s)$

It is then only necessary to remember to apply 1 $kPa \cdot s/m = 1 \times 10^6\ kg/(m^2 \cdot s)$.

Problem D: Electricity

In this problem the units of power and voltage are not explicitly given, but they are clearly implied by the quantities required to be found and used in the calculation (respectively). Specifically, in SI units the power will be reported in watts (W), and the voltage will be specified in volts (V). Current was already specified in the problem statement in amperes or 'amps' (A). Of course, suitable prefixes can also be inserted (MW, kV, mA, *etc*.).

Here the relation between units is not obvious until we recall the definition of volts and amperes in terms of coulombs (C), namely V = J/C, and A = C/s.

Then, on starting with the premise that $P = V\,I$, we have W = J/s = V·A = (J/C)·(C/s).

As the units on either side of the formula are consistent, we have greater confidence that the formula is correct.

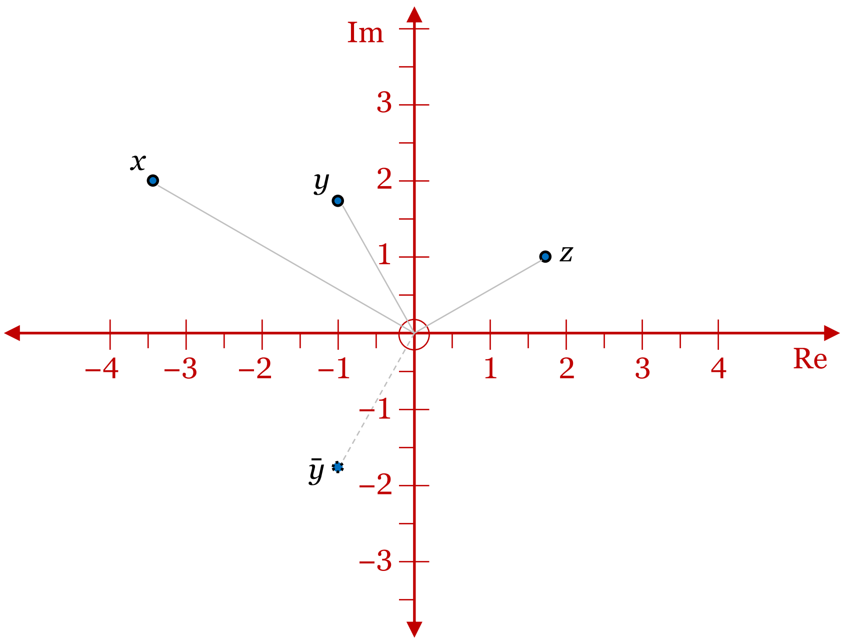
Tactic 4: Use physical knowledge.

Problem A: Division

Not applicable.

Problem B: Complex numbers

The complex numbers can be plotted on the complex plane, on an 'Argand diagramme'. This will be informative if we recall the rules for dividing complex numbers in polar form, namely that the magnitudes obey the normal rules for division, and for the angles (measured anticlockwise from the positive-real axis) the divisor is subtracted from dividend to obtain the quotient.



$x = -2\sqrt{3}+2i$; $y = -1+i\sqrt{3}$; $\bar{y} = -1-i\sqrt{3}$; $z = \sqrt{3}+i$.

Without computing exact values, the proposed answer cannot be 'confirmed', but it certainly looks plausible. The angle between the lines joining points x and y to the origin seems of similar magnitude to the angle made by z with the positive-real axis. It is more difficult to assess 'by inspection' whether the distance of x from the origin could be obtained as the product of the distances of y and z from the origin — but it at least looks like a reasonable possibility.

Problem C: Ultrasonography

Fortunately there are other physical values to compare with in this problem statement. It can be anticipated that velocities in bodily tissues will all have similar orders of magnitude to one another. Likewise densities, and likewise acoustic impedances.

On this basis we feel reassured about the answers proposed for muscle and the lens of the eye, above. We may still lack confidence in the answer proposed for bone above, which indicates an acoustic impedance much larger than any of the other values.

However, we should recall that in ultrasonography, it is not practically possible to obtain images of the brain, due to shielding by the skull, and we should also recall that in B-scan (brightness-mode) images the brightest features are bones such as the vertebræ. This reminds us that — aside from air interfaces — bones are the natural materials most reflective of ultrasound in the body, *i.e.* with the lowest transmission, and thus the acoustic impedance is expected to be substantially different (larger or smaller) in magnitude than the acoustic impedances of other body tissues. With diligent study we would probably remember that the acoustic impedance of bone is higher than the acoustic impedance of other biological materials. With a lot of practice, we might even remember the approximate magnitude of the acoustic impedance of bone: although this would not be expected by the examiners, and should not be a focus of study, it would nonetheless be useful in this specific problem.

For many other problems the problem statement would not provide comparison values. Often it will be sufficient to merely have a sense of what is plausible. For example, if calculating the velocity of a jumbo jet at 'cruising' speed, low values at or under 1 m/s (or 4 km/h) should be immediately ruled out, as these should readily be recognised as walking speed, and values at or above 10^8 m/s (or 10^8 km/h) should also be immediately ruled out, as these would be approaching the speed of light. With a bit of thought, one might remember that it takes roughly one hour to fly the roughly 1000 km between Melbourne and Sydney, and thus deduce that a typical cruising speed might be of order 10^3 km/h.

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Problem D: Electricity

Coincidentally, 10 A is the rating of most domestic fuses, and correspondingly 2400 W is the maximum power consumption rating for domestic appliances that can be plugged into the regular power points (GPO's, general purpose outlets); electric ovens and other such equipment typically run off separate circuits with higher ratings.

Electric kettles, portable room heaters, and toasters are often rated at 2400 W power consumption (10 A current draw), and this might be familiar to some students. Students might also recall the conventional power ratings of incandescent lights (*e.g.* 40 W, 60 W, 75 W, 100 W), which still are printed on the packaging of modern lamps to suggest an equivalent brightness.

Care must be taken to avoid confusion. Some items with very low power draw (*e.g.* modern LED lamps at ~5 W) might incorrectly suggest that 2400 W is implausible. Conversely, the power output of speakers (especially cheaper speakers or sound systems) is sometimes reported as 'peak' values (PMPO), which are misleading insofar as they cannot be directly compared to the usual (RMS) power values.

Tactic 5: Use an alternative method of solution.

Problem A: Division

An obvious check would be to evaluate the problem on an electronic calculator.

Another approach is to recognise that 16 = 2×8 = 4×4 = ... Reducing the divisor to its factors allows a sequence of divisions by single-digit numbers, which does not require long division.

Thus:

$3141 / 16 = 3141 / (2\times8) = (3141 / 2) / 8) = (1570\ ^1/_2) / 8 = (1570 + {}^1/_2) / 8$
$= 196\ ^1/_4 + {}^1/_{(2\times8)} = 196\ ^{(4+1)}/_{16} = 196\ ^5/_{16}$.

$3141 / 16 = 3141 / (4\times4) = (3141 / 4) / 4) = (785\ ^1/_4) / 4 = (785 + {}^1/_4) / 4$
$= 196\ ^1/_4 + {}^1/_{(4\times4)} = 196\ ^{(4+1)}/_{16} = 196\ ^5/_{16}$.

Thereby providing positive confirmation that the proposed answer is correct.

Problem B: Complex numbers

Two approaches can be used to independently check the answer.

One of these is to continue the graphical determination in a quantitative manner. That is, to carefully plot the data, and then physically measure the angles and distances from the origin.

The second is to take a more algebraic approach to the representation in polar form. To do this we recall that any complex number can be fully described by its magnitude (r) and phase (θ), which are equivalent to the distance from the origin and the angle from the positive-real axis, respectively, when plotted in the complex plane. This may be written in several equivalent forms, namely:

$r\ e^{\theta i} = r\ [\cos(\theta) + i \sin(\theta)] = r \operatorname{cis}(\theta)$.

Clearly the real part (a) of this is equal to $r \cos(\theta)$, while the imaginary part (b) of this is equal to $r \sin(\theta)$. By geometrical or algebraic arguments we can also show that $r = \sqrt{a^2 + b^2}$ and $\theta = \operatorname{atan}(a/b) = \tan^{-1}(b/a)$. When the inverse of the tangent function is used, care must be taken to choose the angle pertaining to the correct quadrant.

Thus:

$x = -2\sqrt{3}+2i = \sqrt{12 + 4} \operatorname{cis}[\tan^{-1}(-1/\sqrt{3})] = 4 \operatorname{cis}(150°)$;

$y = -1+i\sqrt{3} = \sqrt{1 + 3} \operatorname{cis}[\tan^{-1}(-\sqrt{3})] = 2 \operatorname{cis}(120°)$;

$\bar{y} = -1-i\sqrt{3} = \sqrt{1 + 3} \operatorname{cis}[\tan^{-1}(\sqrt{3})] = 2 \operatorname{cis}(240°)$;

$z = (4/2) \operatorname{cis}(150° - 120°) = 2 \operatorname{cis}(30°) = 2 \cos(30°) + 2i \sin(30°) = 2\ ({}^{\sqrt{3}}/_2) + 2i\ ({}^1/_2) = \sqrt{3}+i$.

Thereby providing positive confirmation that the proposed answer is correct.

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Problem C: Ultrasonography

For these basic multiplications and divisions there are no obvious alternative solution methods.

Problem D: Electricity

For this basic multiplication there are no obvious alternative solution methods.

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