Methods for Factorising Quadratics

This is a long document, but do not let that intimidate you. It is long for a few reasons:

- *many alternative methods are described;*
- *examples are provided for each method —*
  - *the same four examples (denoted **A**, **B**, **C** & **D**) are used throughout most of the document, to aid comparison;*
- *to more clearly illustrate the techniques, all steps are included —*
  - *full working is shown, including 'dead-ends' that students might encounter.*

Typically a reader will be able to skim through to the section(s) that they are interested in, and focus on reading just that part of the text.

Content

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1. Common factor

A preliminary step is to look for a common factor. This would generally be the largest integer* that is common to all coefficients present.

Problem 1α: All coefficients of equal magnitude

Factorise $7x^2 - 7x + 7$.

Recommended solution

We immediately see that the three coefficients (+7, −7, and +7) all have the same magnitude.
In other words, they have the same absolute value, as $|+7| = |-7| = |+7|$.
Notice that the respective signs must always be considered an inherent part of the coefficients.

Thus we can take out a factor of +7:

$$7(x^2 - x + 1) \,. \qquad \square$$

Notice that this can be expanded to give the original expression.

Note: it can be shown by application of the quadratic formula (§8, p. 19) that this equation cannot be factorised any further, as it has no ‘real number’ roots. However, ‘taking out’ a common factor is useful in either case.

Alternative

We could alternatively have taken out a factor of −7:

$$-7(-x^2 + x - 1) \,. \qquad \square$$

This is mathematically just as correct as the first factorisation. However, we would normally prefer the first option for a few reasons.

- In the second option, we have to write a minus sign in front of the 7, but in the first option the factor of “+7” does not require the sign to be explicitly written. Stylistically, forms with less to write down are preferred.
- It is not favoured stylistically to begin an expression with a minus sign, if it can be easily avoided. We could rearrange the order of terms inside the parentheses, but that would then mean that the polynomial inside the parentheses would no longer be in ‘standard form’, arranged in order of descending indices.
- It is easier to divide by a positive number (where no signs change) than by a negative number (where all signs must change).

Problem 1β: Multiple common factors

Factorise $30x^2 - 12x + 18$.

Iterative method (multiple small factors)

We might notice immediately that all coefficients are even, so 2 must be a common factor.
Although we can thus write down

$$2(15x^2 - 6x + 9) \,,$$

this is not a complete factorisation. We notice that the new coefficients of the expression within the parentheses have a common factor of 3. Thus we can obtain

$$2(\ 3(5x^2 - 2x + 3)\ ) \,,$$
$$= 6(5x^2 - 2x + 3) \,. \qquad \square$$

Note: it can be shown by application of the quadratic formula (§8, p. 19) that this equation cannot be factorised any further, as it has no ‘real number’ roots. However, ‘taking out’ a common factor is useful in either case.

Systematic pairing method (highest common factor)

List the pairs of factors for each coefficient.

+30: {**+1**, +30}; {**+2**, +15}; {**+3**, +10}; {**+5**, +6};
{−1, −30}; {−2, −15}; {−3, −10}; {−5, −6}.

* As exceptions consider the following adaptations of Problem 1α: $7.5x^2 - 7.5x + 7.5$ and $\pi x^2 - \pi x + \pi$.

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-12: {+**1**, −12}; {+**2**, −6}; {+**3**, −4}; {−1, +12}; {−2, +6}; {−3, +4}.
$+18$: {+**1**, +18}; {+**2**, +9}; {+**3**, +6}; {−1, −18}; {−2, −9}; {−3, −6}.

Note: a systematic way of looking for factors is to try to divide by the integers 1, 2, 3, 4, 5 and so on, as indicated in bold above. Stop when you would 'duplicate' an existing pair of factors: for example, {+**6**, +5} is the same pair as {+**5**, +6}.

For positive coefficients, only the pairs of positive coefficients need to be considered.

This follows a similar convention to that discussed in Problem 1α.

If the coefficient of the first term is positive, as it is here (+30), then look for the highest common factor ("**HCF**"). Here the HCF is +6. Now factorise by this number:

$$6(5x^2 - 2x + 3)\ . \qquad \square$$

If the coefficient of the first term were negative, then there would be two reasonable alternatives: either use the HCF, or use the negative of the HCF.

Systematic division method (highest common factor)

Another approach is to consider division of the coefficients by successive integers.

Division of +30, −12, and +18 by +1? Yields an integer for all three.
Division of +30, −12, and +18 by **+2**? Yields an integer for all three.
Division of +30, −12, and +18 by **+3**? Yields an integer for all three.
Division of +30, −12, and +18 by **+4**? Yields an integer only for one.
Division of +30, −12, and +18 by **+5**? Yields an integer only for one.
Division of +30, −12, and +18 by **+6**? Yields an integer for all three.
Division of +30, −12, and +18 by **+7**? Yields an integer for none.
Division of +30, −12, and +18 by **+8**? Yields an integer for none.
Division of +30, −12, and +18 by **+9**? Yields an integer only for one.
Division of +30, −12, and +18 by **+10**? Yields an integer only for one.
Division of +30, −12, and +18 by **+11**? Yields an integer for none.
Division of +30, −12, and +18 by **+12**? Yields an integer only for one.

Stop when the magnitude reaches that of the smallest coefficient.

Division of any integer by +1 always yields the same integer, so this step can be omitted!

Looking through the above list, the HCF is clearly +6, so this should be factorised out of the expression:

$$6(5x^2 - 2x + 3)\ . \qquad \square$$

2. Factorising quadratics is the reverse of expansion

To factorise a quadratic polynomial, perform the reverse of expansion.

a) Monic case

Consider the monic* polynomial obtained by expanding:

$$(1x + p)(1x + q)\ ,$$

namely

$$1x^2 + (q + p)x + pq\ . \qquad \square$$

For this case, the coefficient of the middle term of the expansion is the sum of p and q, while the last term of the expansion is the product of p and q.

It is important to remember again that the signs preceding p and q in the parentheses should be considered an inherent part of their respective values.

Another way of thinking about this is to allow that p or q (or both) can take negative values.

* "Monic" means that the leading coefficient (*i.e.*, the coefficient of the highest power of x, namely x^2 for a quadratic) is equal to 1.

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b) Non-monic case

Consider a general quadratic:

$$(mx + p)(nx + q) .$$

The expansion of this is

$$mnx^2 + (mq + np)x + pq .$$ □

For this case, the last term of the expansion is still the product of p and q, as before. However, the coefficient of the middle term of the expansion is no longer simply the sum of p and q, rather each value is 'weighted' by m or n before being added. Furthermore, the coefficient of the first term of the expansion is no longer simply one, but has become the product of m and n.

3. Factorising by trial-and-error ("guess-and-check")

A common method for factorising quadratics is often called "trial-and-error" * — or, equivalently, "guess-and-check". This has some connotations of choosing 'at random' among possible values, which is apropos when the person performing the factorisation is relatively inexperienced.

On the other hand, when the person performing the factorisation is more experienced, then heuristics are used to speed up the process of obtaining a solution, identifying combinations of values that seem more likely to yield the solution, and avoiding combinations of values that seem less likely to yield the solution.

Problem 3A: Monic quadratic with few factors to consider

Factorise $x^2 - 8x + 7$.

Whether these are written out or not, the process always starts by at least considering the pairs of factors of the constant (+7 in this example).

$+7$: $\{+1, +7\}$; $\{-1, -7\}$.

The product of any of these pairs must be +7, so the factorisation must be built from one of these pairs.

Try the first pair of factors:

$$(x + 1)(x + 7) .$$

This expands to

$$x^2 + 8x + 7 ,$$

which is almost the same as the original expression, except for the sign of the middle coefficient. This indicates that the signs of the constants in the parentheses should be swapped. That is exactly what is achieved by choosing instead the second pair of factors from the list above:

$$(x - 1)(x - 7) ,$$

which expands to

$$x^2 - 8x + 7 ,$$

as desired. So the solution is

$$(x - 1)(x - 7) .$$ □

Note: another mathematically valid solution would be $(-x + 1)(-x + 7)$. However, by convention this solution is not preferred. There are two reasons for that. First, in the original expression x^2 has a coefficient of +1, and it is easier — *i.e.*, less work — to consider only the pair of factors {+1, +1} for this, and ignore the alternative pair of factors {−1, −1}; this is valid, because all combinations of signs are considered for the factors of the constant term. The second reason is simply that it is stylistically undesirable to begin an expression with a negative sign where another alternative exists. Of course, the second solution could equivalently be written $(1 - x)(7 - x)$, but this does not satisfy the preference to write polynomials in order of descending powers.

* For example, this approach is emphasised in the mathcentre's workbook http://www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-factorisingquadratics.pdf

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Problem 3B: Monic quadratic with many factors to consider

Factorise $x^2 + 10x - 24$.

Again, consider pairs of factors of the constant term (−24).

−24: {+1, −24}; {+2, −12}; {+3, −8}; {+4, −6};
{−1, +24}; {−2, +12}; {−3, +8}; {−4, +6}.

Suppose we try the pair {+3, −8} first:

$(x + 3)(x - 8) = x^2 - 5x - 24$.

The expansion of this trial solution has a coefficient of x that is not only the wrong value, but even the wrong sign (−5 instead of +10). That is because −8 has a larger magnitude than +3. Equivalently, using absolute values: $|-8| > |+3|$.

It is easy to get the correct positive sign required here by ensuring that the positive term in the pair of factors has a larger magnitude than the negative term — this is true of all four pairs in the *second row* of the list above, so we should focus on those.

Suppose we now try the pair {−3, +8}:

$(x - 3)(x + 8) = x^2 + 5x - 24$.

Now the sign of the middle term in the expansion is correct, but the magnitude is too small. In fact, notice that the magnitude of the new coefficient (+5) is the same as that obtained in the previous attempt (−5) that used the same pair of factors except for flipping of the signs.

Perhaps we should try {−4, +6}, as four and six sum to the desired figure of ten:

$(x - 4)(x + 6) = x^2 + 2x - 24$.

No, this was a bad choice. The new coefficient (+2) is even further away from the desired value (+10) than in the previous trial (+5). The reason is that the difference in magnitude of −4 and +6 is too small.

Conversely, the pair of factors {−1, +24} seems to have an excessively large difference in their magnitudes, so let's skip that and try instead {−2, +12}:

$(x - 2)(x + 12) = x^2 + 10x - 24$.

Yes, the sum of −2 and +12 is +10, so this leads to the correct solution.

The final answer is thus:

$(x - 2)(x + 12)$. □

Problem 3C: Non-monic quadratic with few factors to consider

Factorise $2x^2 + 5x - 3$.

Now consider factors of both the first coefficient (+2) and the last coefficient (−3).

+2: {+1, +2}; {−1, −2}.

−3: {+1, −3}; {−1, +3}.

For positive coefficients of x^2, only the pairs of positive coefficients need to be considered. (For the third coefficient all pairs should be considered.)

Ignoring the pair {−1, −2} above follows the same convention as mentioned when discussing Problem 3α. It is not wrong to include {−1, −2} in the considerations, but it creates more permutations to work through, and may lead to reporting of a stylistically non-preferred solution (albeit mathematically valid).

There are *four* possible permutations to consider here, as the *sequence* of factors within the second pair matters.

Let us try the first pairs in the order given, namely {+1, −3}:

$(+1x + 1)(+2x - 3) = 2x^2 - 1x - 3$.

Of course, the first and third terms are correctly recovered when expanding this trial solution, but the middle term does not match the original expression, so this must not be the solution sought.

Perhaps we'll have more luck if the order of {+1, −3} is swapped to {−3, +1}?

$(+1x - 3)(+2x + 1) = 2x^2 - 5x - 3$.

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This is looking better; now everything matches the original expression except the sign of the middle coefficient in the expansion.

For monic quadratics we found by experience (Problem 3B) that swapping the signs of the constant terms in the parentheses resulted in the sign of the middle term of the expansion likewise flipping. Thus we shall try {+3, −1}:

$$(+1x + 3)(+2x - 1) = 2x^2 + 5x - 3 .$$

Hence,

$$(x + 3)(2x - 1) .$$ □

must be the desired answer.

Problem 3D: Non-monic quadratic with many factors to consider

Factorise $56x^2 - 158x + 45$.

Pairs of factors should be found for +56 and +45.

+56: {+1, +56}; {+2, +28}; {+4, +14}; {+7, +8};
{−1, −56}; {−2, −28}; {−4, −14}; {−7, −8}.

+45: {+1, +45}; {+3, +15}; {+5, +9};
{−1, −45}; {−3, −15}; {−5, −9}.

For +56 there are four pairs of factors to consider. For +45 there are six pairs of factors to consider, but the *sequence* of the factors within each pair matters. Note that 4×6×2 = 48, so there are *forty-eight* distinct permutations to be considered. This is quite daunting!

The sign of the middle coefficient (−158) is negative, so immediately we know that this can only be obtained by using the pairs {−1, −45}, {−3, −15}, or {−5, −9}. Thus we have swiftly halved the number of permutations by half (to twenty-four) with this simple observation.

The magnitude of the middle coefficient may seem large, but is better described as moderate in comparison to both the first and last coefficients (158/56 ≈ 2.82, and 158/45 ≈ 3.51; 158 is much smaller than 56×45 = 2520).

Let’s therefore try starting with mid-range pairs: {+4, +14} and {−3, −15}.

$$(+4x - 3)(+14x - 15) = 56x^2 - 102x + 45 .$$

This is pretty close, but the magnitude of −102 is not large enough. Let’s try swapping the constants inside the parentheses.

$$(+4x - 15)(+14x - 3) = 56x^2 - 222x + 45 .$$

Now the problem is that the magnitude of −222 is too large.

It is more systematic to just change one of the pairs.

Let’s now try {+4, +14} and {−5, −9}.

$$(+4x - 5)(+14x - 9) = 56x^2 - 106x + 45 .$$

Swapping the second pair will presumably increase the magnitude of the middle coefficient in the expansion as it did before:

$$(+4x - 9)(+14x - 5) = 56x^2 - 146x + 45 .$$

This is so close, but not quite there. Let’s check what happens with {+4, +14} and {−1, −45}.

$$(+4x - 1)(+14x - 45) = 56x^2 - 194x + 45 .$$

The other combination involves −45×14 and will clearly be too big.

There is no permutation involving {+4, +14} that yielded the desired answer!

Using {+4, +14} and {−9, −5} gave the nearest match so far, so perhaps we can improve on that with an adjustment to {+7, +8} and {−9, −5}.

$$(+7x - 9)(+8x - 5) = 56x^2 - 107x + 45 .$$

This has made things worse. Although the combination of {+7, +8} and {−5, −9} is yet to be checked, let's first try making an adjustment to the above in the opposite direction. That is, trying {+2, +28} and {−9, −5}.

$$(+2x-9)(+28x-5) = 56x^2 - 262x + 45\ .$$

The above multiplies −9 by +28 and results in too large a magnitude. That can be reduced by swapping the constants inside the parentheses.

$$(+2x-5)(+28x-9) = 56x^2 - 158x + 45\ .$$

Finally we have an expansion that matches the original expression.

To reiterate, the original expression can be factorised to

$$(2x-5)(28x-9)\ .$$ □

Of the reduced set of twenty-four candidate factorisations, we completed expansions for eight. Sometimes we might be lucky and be able to guess the factorisation on our first attempt. Other times we might exhaust almost all of the possibilities before finally obtaining the correct factorisation.

4. Factorising by trial-and-error, using a cross layout

Given the need to consider many permutations, it might be considered tedious to write each candidate factorisation out in full, and to expand each term. As seen in the previous section, once factors have been identified from the first and last coefficients, all of the attention shifts to ensuring that the middle term of the expansion can match that in the original expression.

An alternative is to write this out using a cross layout, either in a large schematic presentation or in a compact table.

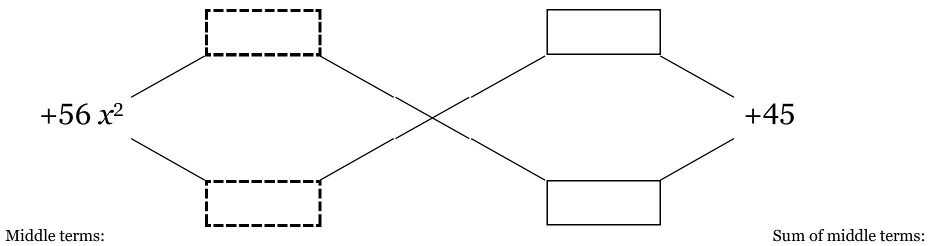
a) Large schematic presentation

In the following, one sample problem is solved by factorising with the aid of a cross layout, presented in the form of a large expanded schematic diagramme. This depiction of a cross layout has the advantage of visually emphasising the "cross" relationship, although it takes up more space, and is less suited for systematic consideration of pairs of factors.

Problem 4D (*schematic*): Non-monic quadratic with many factors

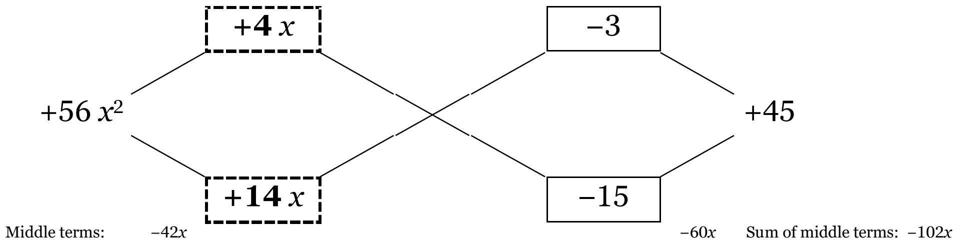
Factorise $56x^2 - 158x + 45$.

Solving as in Problem 3D we would start by entering pairs of factors into the template below.



Specifically, pairs of factors of $+56x^2$, are entered into the two left-hand boxes, and pairs of factors of +45 are entered into the two right-hand boxes. The diagonal lines at far left and far right remind us of the origin of those factors. The middle term ($-158x$) should be obtained by multiplying the pairs of factors crosswise, and then adding the two results.

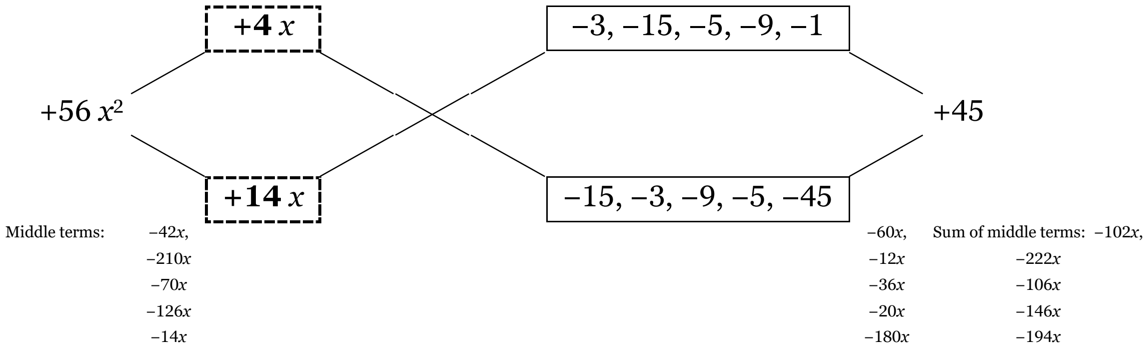
For example, considering the first combination yields the following.



Clearly $-102x$ does not match the target of $-158x$ for the middle term in this problem.

So what should we do now? Draw up the entire template again? That's a bit tedious, and takes up quite a bit of space. Should we just cross out the old guesses, and write some new guesses next to them? That can become hard to read. (And similarly if we write in pencil, then erase and rewrite many times.)

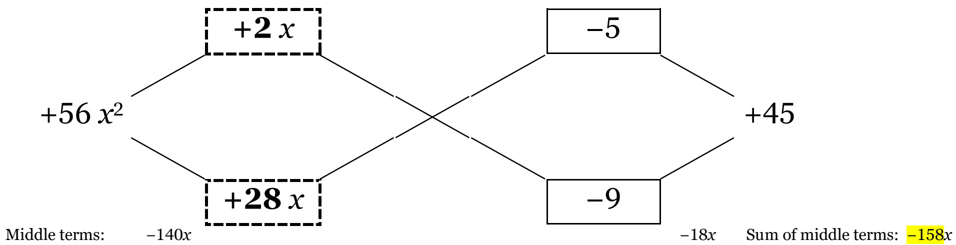
Perhaps we could make the template larger to begin with, in anticipation of this, and then work through the pairs of factors systematically.



It might be easy to get confused working with so many numbers, so it could help to strike out each unsuccessful combination before proceeding to the next attempt.

As before, none of these options yields the target of $-158x$ for the middle term. So we must consider a different pair of factors for $+56x^2$.

Eventually (*confer* pp. 9f.), we can arrive at



From whence we see that the original expression can be factorised to

$$(2x - 5)(28x - 9) .$$

□

b) Compact tabular presentation

In the following, the previous problems are solved by factorising with the aid of a cross layout that is presented in the form of a compact table. This has the advantage of not taking much space, and also allowing pairs of factors to be systematically considered. On the other hand, the "cross" relationship is not visually emphasised as much as in some other presentations (*vide supra*).

Problem 4A: Monic quadratic with few factors to consider

Factorise $x^2 - 8x + 7$.

Proceeding as in Problem 3A, note the pairs of factors of the constant (+7).

+7: {+1, +7}; {−1, −7}.

The coefficient of x^2 is clearly to be obtained from {+1, +1}.

We write in a kind of rough table:

+1 :	+1
+1 :	+7
Middle coefficient:	+8

Multiplying diagonally and adding the results gives +1×+7=+7 (top left and bottom right) and +1×+1=+1 (bottom left and top right); the sum of these is +8, rather than the desired −8.

Thus to the original table we append an alternative pairing. We may find it helpful strike out the unsuccessful candidates as we go.

+1 :	~~+1~~	−1
+1 :	~~+7~~	−7
Middle coefficient:	+8	−8

Multiplying diagonally and adding the results gives +1×−7=−7 and +1×−1=+−1; the sum of these is the desired −8. Hence, the solution must be

$$(x-1)(x-7)\ .$$ □

Problem 4B: Monic quadratic with many factors to consider

Factorise $x^2+10x-24$.

Solving as in Problem 3B we would work through a table as below

+1 :	~~+3~~	~~−3~~	~~−4~~	−2
+1 :	~~−8~~	~~+8~~	~~+6~~	+12
Middle coefficient:	−5	+5	+2	+10

and find the answer to again be

$$(x-2)(x+12)\ .$$ □

It might be helpful to keep track of the computed coefficient of the middle term just below the tabulation, as shown above.

Problem 4C: Non-monic quadratic with few factors to consider

Factorise $2x^2+5x-3$.

Solving as in Problem 3C we would work through a table as below

+1 :	~~+1~~	~~−3~~	+3
+2 :	~~−3~~	~~+1~~	−1
Middle coefficient:	−1	−5	+5

and again find the answer to be

$$(x+3)(2x-1)\ .$$ □

Problem 4D: Non-monic quadratic with many factors to consider

Factorise $56x^2-158x+45$.

Solving as in Problem 3D we would start by working through a table as below

+4 :	~~−3~~	~~−15~~	~~−5~~	~~−9~~	~~−1~~
+14 :	~~−15~~	~~−3~~	~~−9~~	~~−5~~	~~−45~~
Middle coefficient:	−102	−222	−106	−146	−194

There is no permutation involving {+4, +14} that yielded the desired answer!

To move on from the pair {+4, +14} to {+7, +8} we have two ways of laying this out. The quickest way would be to prefix the new numbers to the original tabulation, as in

+7 :	**+4 :**	−3	−15	−5	−9	−1
+8 :	**+14 :**	−15	−3	−9	−5	−45
	Middle coefficient:	−102	−222	−106	−146	−194
Middle coefficient:						

This can get a bit messy, and harder to keep track of workings and sums. It is also awkward if unsuitable candidates had been crossed out and need to be considered again.

Thus, all around it seems better just to set up a new table (or tables).

+7 :	−9
+8 :	−5
Middle coefficient:	−107

+2 :	−9	−5
+28 :	−5	−9
Middle coefficient:	−262	−158

From whence we see that the original expression can be factorised to

$(2x - 5)(28x - 9)$. □

5. Factorising using the PSF method (ACE method)

a) Conventional implementation of the PSF method (ACE method)

i. Derivation

Recall in section 3 we saw that the expansion of a general quadratic:

$(mx + p)(nx + q)$

is

$mnx^2 + (mq + np)x + pq$. □

The three coefficients of the expansion are mn, $(mq + np)$, and pq.

Suppose we multiply the first coefficient (mn) by the last (pq): we would obtain $mnpq$. Now notice that the individual terms comprising the second coefficient (that is, mq and np) are factors of $mnpq$.

ii. Method

The above derivation motivates the PSF method* (also known as the ACE method†), which involves the following steps:

1. Obtain the product of the first coefficient and the last coefficient.
2. Find pairs of factors of the product found in step 1.
3. Select the pair of factors found in step 1 whose sum is equal to the middle coefficient.
4. Rewrite the middle term in x as the sum of two terms in x with coefficients equal to the factors found in step 3.
5. Factorise by two successive pairing operations.

Ensure, as always, to always treat signs as inherent elements of the factors, sums, products, or coefficients.

* PSF is an abbreviation for "product, sum, and factors".

† For example, this approach is emphasised in Worksheet 2.6 of the Macquarie University Mathematical Skills course http://maths.mq.edu.au/numeracy/web_mums/module2/Worksheet26/module2.pdf
It is named "ACE" as a (somewhat cryptic!) mnemonic: pronounce "ACE" as "eh" and "see", representing the letters "a" and "c" that are often used to denote the first and last coefficient of a quadratic expression.

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The benefit of this approach is greatest for non-monic expressions. That is because for monic expressions the coefficient of x^2 is (by definition) equal to +1, and multiplication by +1 (as in step 1, above) does not change the number for which factors are to be found.

Problem 5A: Monic quadratic with few factors to consider

Factorise $x^2 - 8x + 7$.

The product of +1 and +7 is +7. We need to find factors of this number (as in Problem 3A/4A).

There are a couple of pairs of factors of +7, but the pair that sums to −8 is {−1, −7}.

Hence we can write

$$x^2 - 8x + 7$$
$$= x^2 + (-7 - 1)x + 7$$
$$= x^2 - 7x - 1x + 7 .$$

Look for pairs to factorise.

Note: $x(x - 7) - 1x + 7$ is not helpful, as there is no way to factorise the second, remaining pair of terms.

Write the terms in a slightly different order

$$= x^2 - 1x - 7x + 7 ,$$

and then factorise in pairs

$$= x^2 - 1x - 7x + 7$$
$$= x(x - 1) + 7(-x + 1) .$$

This is better, but the next step is not apparent because the two expressions in parentheses are not the same. That is easy to fix by taking out a further factor of −1 from the second set of terms:

$$= x(x - 1) - 7(x - 1) .$$

How many lots of $(x - 1)$ does this represent? The first set of terms provides x lots, and the second set of terms subtracts 7 lots, so overall there are $(x - 7)$ lots of $(x - 1)$. This is a wordy way of expressing the mathematical formula

$$= (x - 7)(x - 1) ,$$ □

as found before.

Problem 5B: Monic quadratic with many factors to consider

Factorise $x^2 + 10x - 24$.

The product of +1 and −24 is −24. We need to find factors of this number (as in Problem 3B/4B).

There are several pairs of factors of −24, but only one pair of factors sums to +10, namely {−2, +12}.

Hence we can write

$$x^2 + 10x - 24$$
$$= x^2 + (-2 + 12)x - 24$$
$$= x^2 - 2x + 12x - 24 .$$

Look for pairs to factorise.

$$= x(x - 2) + 12(x - 2) .$$

Fortunately we did not have to rearrange the terms to obtain compatible pairs, and the terms in parentheses are the same in each instance. Hence we can say there are $(x + 12)$ lots of $(x - 2)$, in other words

$$= (x + 12)(x - 2) ,$$ □

as found before.

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In Problem 3B and Problem 4B the answer was expressed as $(x-2)(x+12)$, but of course for multiplication the order is immaterial.

Problem 5C: Non-monic quadratic with few factors to consider

Factorise $2x^2 + 5x - 3$.

The product of +2 and −3 is −6. We need to find factors of this number (different from Problem 3C/4C).

There are a few pairs of factors of −6.

−6: {+1, −6}; {+2, −3}; {−1, +6}; {−2, +3}.

Only one of these pairs sums to +5, namely {−1, +6}.

Hence we can write

$2x^2 + 5x - 3$

$= 2x^2 + (-1 + 6)x - 3$

$= 2x^2 - 1x + 6x - 3$.

Look for pairs to factorise.

$= x(2x - 1) + 3(2x - 1)$.

Fortunately we did not have to rearrange the terms to obtain compatible pairs, and the terms in parentheses are the same in each instance. Hence we can now obtain

$= (x + 3)(2x - 1)$, □

as found before.

Problem 5D: Non-monic quadratic with many factors to consider

Factorise $56x^2 - 158x + 45$.

The product of +56 and +45 is +2520. We need to find factors of this number (different from Problem 3D/4D).

There is a very large set of pairs of factors of +2520 (there are forty-eight pairs). We could obtain these following the system described in Problem 1β. We could alternatively consider separately the factors of +56 and +45 (listed on p. 6) — which must also be factors of +2520 — and combinations thereof.

The most notable disadvantage of the ACE and PSF methods is the potential need to consider factors of largish numbers, for which factors may be troublesome to find without the aid of an electronic calculation aid.

To save time, before writing down any pairs of factors we notice that the sum of the required pair must be −158. This tells us that only pairs containing two negative values need to be considered.

Clearly there is no way to obtain a sum of −158 from two positive numbers.

Now only twenty-four pairs need to be considered. We start off

+2520: {−1, −2520}; {−2, −1260}; ....

These pairs have sums with excessively large magnitudes. Perhaps, to save time, we can try skipping a few pairs.

+2520: {−1, −2520}; {−2, −1260}; .... ; {−14, −180}; {−15, −168}; {−18, −140}; ....

Finally, we see that the pair of factors {−18, −140}* sums to −158.

Although there are still several more pairs in the sequence, not to mention the pairs that were skipped, we are looking to find just one unique pair of factors. Now that we have found it, we do not need to check any other pairs.

Hence we can write

$56x^2 - 158x + 45$

$= 56x^2 + (-18 - 140)x + 45$

* Referring back to Problem 3D (pp. 6f.), this pair can also be generated by combining {+2, +28} and {−5, −9}, which are factors of +56 and +45, respectively. The alternative combination yields {−10, −252} — one of the pairs that was skipped over in the above discussion.

$= 56x^2 - 18x - 140x + 45$.

Look for pairs to factorise. With these larger numbers it is not so obvious. If necessary, this could be done iteratively (*confer* Problem 1β).

$= 2x(28x - 9) + 5(-28x + 9)$

$= 2x(28x - 9) - 5(28x - 9)$.

Hence we can obtain

$= (2x - 5)(28x - 9)$, □

as found before.

b) Alternative implementation of the PSF method (ACE method)

i. Derivation

Another way of implementing the PSF method is based on temporary multiplication of every term by the leading coefficient, then factorisation, then division to recover the original scaling.

As before, recall that the expansion of a general quadratic:

$(mx + p)(nx + q)$

was shown in section 3 to be

$mnx^2 + (mq + np)x + pq$.

The three coefficients of the expansion are mn, $(mq + np)$, and pq.

Now we can simultaneously multiply and divide the entire expression by the leading coefficient, $\times(mn)/(mn)$, which is valid because it is equivalent to multiplying by 1.

$mnx^2 + (mq + np)x + pq$

$= [mnx^2 + (mq + np)x + pq] \times (mn)/(mn)$

$= (mn)[mnx^2 + (mq + np)x + pq]/(mn)$

$= [(mn)^2x^2 + (mn)(mq + np)x + mnpq]/(mn)$

$= [(mn)^2x^2 + (m^2nq + mn^2p)x + mq\times np]/(mn)$

$= (mnx + np)(mnx + mq)/(mn)$

$= (mnx + np)/n \quad \times \quad (mnx + mq)/m$

$= (mx + p) \quad \times \quad (nx + q)$

$= (mx + p)(nx + q)$. □

A distinctive feature is the intermediate step in which the factorised terms both have the same coefficient for x (highlighted above).

ii. Method

The above derivation motivates an alternative algorithm for the PSF method (a.k.a. the ACE method), which involves the following steps:

1. Obtain the product of the first coefficient and the last coefficient.
2. Find pairs of factors of the product found in step 1.
3. Select the pair of factors found in step 1 whose sum is equal to the middle coefficient.
4. Rewrite the middle term in x as the sum of two terms in x with coefficients equal to the factors found in step 3.
5. Simultaneously multiply and divide the entire expression by the leading coefficient.
6. Factorise by forming two separate terms that each contain x with original leading coefficient, with the constants corresponding to the pair of factors found at step 3.
7. Simplify.

Notice that the first four steps are identical to the conventional PSF method (see p. 10). Step 4 could optionally be skipped here.

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Problem 5C (*alternative*): Non-monic quadratic with few factors

Factorise $2x^2 + 5x - 3$.

The product of +2 and −3 is −6. We need to find factors of this number (different from Problem 3C/4C, but the same as on p. 12).

There are a few pairs of factors of −6.

−6: {+1, −6}; {+2, −3}; {−1, +6}; {−2, +3}.

Only one of these pairs sums to +5, namely {−1, +6}.

Hence we can write

$2x^2 + 5x - 3$

$= 2x^2 + (-1 + 6)x - 3$

$= 2x^2 - 1x + 6x - 3$.

Now multiply by 2/2:

$= 2\,[2x^2 - 1x + 6x - 3] / 2$

$= [4x^2 - 2x + 12x - 6] / 2.$

The original leading coefficient was +2, so seek to a factorisation in the form $(2x \pm \ldots)(2x \pm \ldots)$, in which the blanks shall be filled by the pair of factors found earlier, namely {−1, +6}. Thus

$= (2x - 1)(2x + 6) / 2.$

Finally, simplify

$= (2x - 1)(x + 3)$

$= (x + 3)(2x - 1)$, □

as found before.

With experience, the steps shown in grey could be omitted from the working.

Problem 5D (*alternative*): Non-monic quadratic with many factors

Factorise $56x^2 - 158x + 45$.

The product of +56 and +45 is +2520. We need to find factors of this number (different from Problem 3D/4D, but the same as on p. 12).

As mentioned previously, there is a very large set of pairs of factors of +2520 (there are forty-eight pairs). Eventually we would find that the pair of factors {−18, −140} sums to −158.

Hence we can write

$56x^2 - 158x + 45$

$= 56x^2 + (-18 - 140)x + 45$

$= 56x^2 - 18x - 140x + 45$.

Now multiply by 56/56:

$= 56\,[56x^2 - 18x - 140x + 45] / 56$

$= [3136x^2 - 1008x - 7840x - 6] / 56.$

The original leading coefficient was +56, so seek to a factorisation in the form $(56x \pm \ldots)(56x \pm \ldots)$, in which the blanks shall be filled by the pair of factors found earlier, namely {−18, −140}. Thus

$= (56x - 18)(56x - 140) / 56.$

Finally, simplify. It's not necessarily obvious how to complete this in a single step; it's perfectly acceptable to perform the simplification in several steps. For example, notice that all of the parenthesised terms are even, and that 4 is a factor of 56.

$= (56x - 18)(56x - 140) / (14\times2\times2)$

$= [(56x - 18)/2 \times (56x - 140)/2\,] / 14$

$= [(28x - 9)(28x - 70)] / 14$

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$= (28x - 9) \times (28x - 70)/14$
$= (28x - 9) \times (2x - 5)$, □

as found before.

With experience, the steps shown in grey could be omitted from the working.

6. Factorising by 'completing the square' then using the 'difference of two squares'

It is possible to factorise a quadratic polynomial by '**completing the square**' and then applying the '**difference of two squares**'.

For a monic quadratic polynomial

$x^2 + bx + c$
$= x^2 + bx + (b/2)^2 - (b/2)^2 + c$

Notice that the terms in purple cancel each other out, and the highlighted terms comprise a 'perfect square'.

$= (x + b/2)^2 - (b/2)^2 + c$
$= (x + b/2)^2 - (b/2)^2 + 4c/2^2$
$= (x + b/2)^2 - (b^2 - 4c)/2^2$
$= (x + b/2)^2 - (\sqrt{b^2 - 4c}/2)^2$
$= (x + b/2 + \sqrt{b^2 - 4c}/2)\ (x + b/2 - \sqrt{b^2 - 4c}/2)$
$= (x + \frac{b + \sqrt{b^2 - 4c}}{2})\ (x + \frac{b - \sqrt{b^2 - 4c}}{2})$. □

For a general quadratic polynomial

$ax^2 + bx + c$
$a[x^2 + (b/a)x + c/a]$
$= a[x^2 + (b/a)x + (b/2a)^2 - (b/2a)^2 + c/a]$

Notice that the terms in purple cancel each other out, and the highlighted terms comprise a 'perfect square'.

$= a[(x + b/2a)^2 - (b/2a)^2 + c/a]$
$= a[(x + b/2a)^2 - (b/2a)^2 + 4ac/(2a)^2]$
$= a[(x + b/2a)^2 - (b^2 - 4ac)/(2a)^2]$
$= a[(x + b/2a)^2 - (\sqrt{b^2 - 4ac}/2a)^2]$
$= a\ (x + b/2a + \sqrt{b^2 - 4ac}/2a)\ (x + b/2a - \sqrt{b^2 - 4ac}/2a)$
$= a\ (x + \frac{b + \sqrt{b^2 - 4ac}}{2a})\ (x + \frac{b - \sqrt{b^2 - 4ac}}{2a})$. □

Notice that the general quadratic polynomial will thus be equal to zero when

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} ,$$

which is the so-called '**quadratic formula**' (§8, pp.19ff.). Given that the quadratic formula is derived by 'completing the square' on the general formula for a quadratic polynomial, completing the square might involve more work than would be needed if we simply applied the quadratic formula.

Problem 6A

Factorise $x^2 - 8x + 7$.

First we complete the square.

$x^2 - 8x + 7$

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$= x^2 - 8\,x + (-8/2)^2 - (-8/2)^2 + 7$
$= (x - 4)^2 - (-4)^2 + 7$
$= (x - 4)^2 - 16 + 7$
$= (x - 4)^2 - 9$.

Now we recognise the above as a difference of two squares.

$= (x - 4)^2 - 3^2$
$= (x - 4 + 3)\,(x - 4 - 3)$
$= (x - 1)\,(x - 7)$, □

as found before.

Problem 6B

Factorise $x^2 + 10x - 24$.

First we complete the square.

$x^2 + 10x - 24$
$= x^2 + 10\,x + (10/2)^2 - (10/2)^2 - 24$
$= (x + 5)^2 - (5)^2 - 24$
$= (x + 5)^2 - 25 - 24$
$= (x + 5)^2 - 49$

Now we recognise the above as a difference of two squares.

$= (x + 5)^2 - 7^2$
$= (x + 5 + 7)\,(x + 5 - 7)$
$= (x + 12)\,(x - 2)$, □

as found before.

Problem 6C

Factorise $2x^2 + 5x - 3$.

First we complete the square.

$2x^2 + 5x - 3$
$= 2[x^2 + 5/2\,x - 3/2]$
$= 2[x^2 + 5/2\,x + (5/4)^2 - (5/4)^2 - 3/2]$
$= 2[(x + 5/4)^2 - (5/4)^2 - 3/2]$
$= 2[(x + 5/4)^2 - 25/16 - 24/16]$
$= 2[(x + 5/4)^2 - 49/16]$.

Now we recognise the above as a difference of two squares.

$= 2[(x + 5/4)^2 - (7/4)^2]$
$= 2\,(x + 5/4 + 7/4)\,(x + 5/4 - 7/4)$
$= 2\,(x + 12/4)\,(x - 2/4)$
$= 2\,(x + 3)\,(x - 1/2)$.

It is looks better to multiply the prefactor of +2 into the terms in the second set of parentheses to avoid writing out any fractions. This yields the same answer as obtained before by other methods

$(x + 3)(2x - 1)$. □

Notice that there were quite a few fractions to deal with, and two different levels of brackets/parentheses.

It's good practice to use distinct brackets, parentheses and braces to distinguish different levels.

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Problem 6D

Factorise $56x^2 - 158x + 45$.

First we complete the square.

$56x^2 - 158x + 45$

$= 56[x^2 - 158/56\,x + 45/56]$

$= 56[x^2 - 158/56\,x + (-158/112)^2 - (-158/112)^2 + 45/56]$

Noting the large numbers already involved, we may try to simplify the fractions before proceeding. It will turn out (see below) that simplifying 158/56 is not really necessary (though not harmful either).

$= 56[x^2 - 79/28\,x + (-79/56)^2 - (-79/56)^2 + 45/56]$

$= 56[(x - 79/56)^2 - (79/56)^2 + 45/56]$

$= 56[(x - 79/56)^2 - 79^2/56^2 + (45\times56)/56^2]$

Even after simplifying the fractions we still have to deal with some large numbers. However, we do not have to evaluate 56^2, as we anticipate the implementation of 'difference of two squares'.

$= 56[(x - 79/56)^2 - 6241/56^2 + 2520/56^2]$

$= 56[(x - 79/56)^2 - 3721/56^2]$.

Even though it is not at all obvious, in principle we should be able to interpret the above as a difference of two squares.

$= 56[(x - 79/56)^2 - (\sqrt{3721}/56)^2]$

The practical problem is to evaluate $\sqrt{3721}$. It can, of course, be computed with an electronic calculator. If we anticipate that the result will be an integer, then we can home in on the value by evaluating some easier results to provide context: $10^2 = 100$, $20^2 = 400$, $30^2 = 900$, $40^2 = 1600$, $50^2 = 2500$, $60^2 = 3600$, $70^2 = 4900$, $80^2 = 6400$, and $90^2 = 8100$. The current problem is closest to $\sqrt{3600} = 60$, and (given that 3600+60+61=3721) it turns out that $\sqrt{3721} = 61$.

$= 56[(x - 79/56)^2 - (61/56)^2]$

$= 56(x - 79/56 + 61/56)\,(x - 79/56 - 61/56)$

$= 56\,(x - 18/56)\,(x - 140/56)$.

This looks very ugly. We would hope to find a nicer way to present the result by selectively multiplying factors of +56 into the terms in the first and second sets of parentheses to avoid writing out any fractions. It will be easier to see which way to do this by simplifying the fractions first.

$= 56\,(x - 18/56)\,(x - 140/56)$

$= 56\,(x - 9/28)\,(x - 35/14)$

$= 56\,(x - 9/28)\,(x - 5/2)$

$= (28\times2)\,(x - 9/28)\,(x - 5/2)$

$= 28\,(x - 9/28) \times 2\,(x - 5/2)$

$= (28x - 9)\,(2x - 5)$. □

This finally yields the same answer as obtained before by other methods.

Notice that there were quite a few fractions to deal with, various inconveniently large numbers, and two different levels of brackets/parentheses. There is a lot of working, and it would be easy for anyone to make a mistake.

7. Roots of a polynomial

a) Properties of the roots

i. Value of a polynomial

The roots are the values of x for which the polynomial's value is zero.

For example,

If $x = 0$, then $x^3 + x^2 = 0$.

Thus, $x = 0$ is a root of $x^3 + x^2$.

If $x = 1$, then $x^2 + x - 2 = 0$.

Thus, $x = 1$ is a root of $x^2 + x - 2$.

The maximum possible number of roots that a polynomial may have is given by its degree. Thus, a quartic (degree 4) may have up to 4 roots — or as few as none; a cubic (degree 3) may have up to 3 roots — or as few as one; a quadratic (degree 2) may have up to 2 roots — or as few as none.

For example, complementing the roots already found above, we find the following.

If $x = -1$, then $x^3 + x^2 = 0$.

If $x = -2$, then $x^2 + x - 2 = 0$.

For these examples there are no more roots. Thus, the cubic here had two roots, as did the quadratic.

An alternative interpretation says that a cubic always has three roots; in such an interpretation we would say that $x = 0$ must be counted twice, as apparent by writing the factorisation as $(x)(x)(x + 1) = 0$. For some other polynomials it is only possible to produce the desired number of roots for this interpretation by resorting to 'imaginary numbers' — an advanced concept that allows the square root of a negative number to be obtained.

ii. The polynomial as a function

We are familiar with many functions, such as the sine function of trigonometry, which takes an input, and gives one output for that input. We would write, for example:

$\sin(30°) = ½$.

Notice that 30° is the input, and ½ is the output.

The cube root is also a function. We could define our own compact name for this common function; say, "r". Then

r(8) = 2 ,

r(27) = 3 ,

r(64) = 4 ,

r(125) = 5 ,

and so on.

Note: r is the name of the function; it is not a variable. Thus we must not try to find "r times 27" or "r times 125" above. Of course, it is still possible to perform operations on functions, but that is written as, say, $2 \times \sin(30°)$, which is equal to 1; or r(125) + 1 , which is equal to 6.

A polynomial can also be treated as a function. Commonly the name of the function is given as "f" (for 'function') or "p" (for 'polynomial'). Let's use "f" here.

When the value of x has not yet been specified, we can have, for example,

$f(x) = x^2 + x - 2$.

At some arbitrary value of x we have, say

$f(10) = 10^2 + 10 - 2 = 108$.

At the roots, we have

$f(1) = 1^2 + 1 - 2 = 0$,

$f(-2) = (-2)^2 + (-2) - 2 = 0$.

iii. Graphing a polynomial

We can also graph a polynomial *. By convention we assign the value of the polynomial to a new variable, y, and plot this on the vertical axis of our graph.

Thus, for the quadratic $x^2 + x - 2$ we would write

$y = x^2 + x - 2$.

The distinction between y and $f(x)$ is that y is a variable, whereas $f(x)$ is a function — whose input is stated to be the variable x.

We saw already that this polynomial has two roots, namely +1 and −2. At each root the value of the polynomial is zero; thus, at each root $y = 0$. Therefore, the roots are the locations of

* This is emphasised by the 'Math Is Fun' website (at 'MathsIsFun.com') at https://www.mathsisfun.com/algebra/factoring-quadratics.html

the x-intercepts, where the curve crosses the x-axis. The y-intercept is quickly seen to be −2 (corresponding to $x = 0$).

This is sufficient for a rough sketch of the polynomial, although for more detail the turning point would also be annotated.

b) From roots to factorisation

Consider a general quadratic polynomial

$$ax^2 + bx + c\,.$$

Suppose the two roots are called P and Q. Then the quadratic polynomial can be factorised as

$$ax^2 + bx + c$$
$$= a(x - P)(x - Q)\,.$$

Note the signs.

It is possible to multiply the prefactor of a into one of the sets of parentheses, such as $(ax - aP)(x - Q)$ or $(x - P)(ax - aQ)$; or even to split it between the two sets, as in $(ax/2 - aP/2)(2x - 2Q)$. Whether this provides a stylistic advantage, or any other advantage, should be determined on a case-by-case basis.

This is true because if we wanted to find the roots of $ax^2 + bx + c$, then we would need to set

$$ax^2 + bx + c = 0\,,$$

and the above statement indicates that this is equivalent to setting

$$a(x - P)(x - Q) = 0\,,$$

and the only ways that this latter equation can be satisfied are if either $x = P$ or $x = Q$.

8. Factorising using the 'quadratic formula' to find roots

a) Application of the quadratic formula

The so-called 'quadratic formula' **always** yields the roots of a quadratic polynomial.

For the general quadratic polynomial

$$ax^2 + bx + c\,,$$

the roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

The derivation was given in §6 (pp. 15f.).

Notice that the "±" sign provides two distinct roots if the term under the square root sign (the "**discriminant**") is greater than zero. If the discriminant is zero, then the two roots are identical. If the discriminant is negative, then there are no 'real' roots (only 'imaginary' ones).

Problem 8A: Monic quadratic with few factors

Factorise $x^2 - 8x + 7$.

The roots are

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\times1\times7}}{2\times1} = \frac{+8 \pm \sqrt{64 - 28}}{2} = \frac{8 \pm \sqrt{36}}{2} = 4 \pm 3 = +1,\ +7.$$

This tells us that a factorisation must be

$$+1(x - (+1))(x - (+7))\,.$$

Simplify this to obtain the same answer as obtained before by other methods

$$(x - 1)(x - 7)\,.$$

□

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Problem 8B: Monic quadratic with many factors

Factorise $x^2 + 10x - 24$.

The roots are

$$x = \frac{-(10) \pm \sqrt{10^2 - 4\times1\times-24}}{2\times1} = \frac{-10 \pm \sqrt{100 + 96}}{2} = \frac{-10 \pm \sqrt{196}}{2} = -5 \pm 7 = -12, +2.$$

This tells us that a factorisation must be

$+1(x - (-12))(x - (+2))$.

Simplify this to obtain the same answer as obtained before by other methods

$(x + 12)(x - 2)$. □

Notice that the long list of pairs of factors that had to be considered in previous methods does not need to be looked at when using the quadratic formula.

Problem 8C: Non-monic quadratic with few factors

Factorise $2x^2 + 5x - 3$.

The roots are

$$x = \frac{-(5) \pm \sqrt{5^2 - 4\times2\times-3}}{2\times2} = \frac{-5 \pm \sqrt{25 + 24}}{4} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4} = -3, +½.$$

This tells us that a factorisation must be

$+2(x - (-3))(x - (+½))$.

Or, equivalently

$+2(x + 3)(x - ½)$.

It is stylistically non-preferred to keep the prefactor of +2 out the front while a non-integer term exists in the parentheses. Multiplying that +2 into the terms in the second set of parentheses yields the same answer as obtained before by other methods

$(x + 3)(2x - 1)$. □

Problem 8D: Non-monic quadratic with many factors

Factorise $56x^2 - 158x + 45$.

The roots are

$$x = \frac{-(-158) \pm \sqrt{(-158)^2 - 4\times56\times45}}{2\times56} = \frac{158 \pm \sqrt{24964 - 10080}}{112} = \frac{158 \pm \sqrt{14884}}{112}$$

$$= \frac{158 \pm 122}{112} = +36/112, +280/112 = +9/28, +2½.$$

As will be seen below, it is advantageous to report these roots as fractions, rather than decimals.

This tells us that a factorisation must be

$+56(x - (+9/28))(x - (+2½))$

$= 56(x - 9/28)(x - 2½)$.

This looks rather ugly. Suppose we deal with the half-term first.

$= 28(x - 9/28)(2x - 5)$.

Now it becomes more obvious how to avoid writing the other fraction. Multiplying the remaining +28 into the terms in the first set of parentheses yields the same answer as obtained before by other methods

$= (28x - 9)(2x - 5)$. □

Notice that the long lists of pairs of factors that had to be considered in previous methods do not need to be looked at when using the quadratic formula.

b) ‘Completing the square’ to obtain the roots

As shown in §6 (pp. 15ff.) it is possible to ‘complete the square’ to obtain a factorisation. In fact, the quadratic formula is derived by ‘completing the square’ on the general formula for a quadratic polynomial. Although it is possible to perform factorisation by ‘completing the square’ to obtain the roots, it involves more work than would be needed if we simply applied the quadratic formula.

Example

Factorise $2x^2 + 5x - 3$.

This is the same as Problem 6C/8C (*vide supra*).

$2x^2 + 5x - 3$
$= 2[x^2 + 5/2\,x - 3/2]$
$= 2[x^2 + 5/2\,x + (5/4)^2 - (5/4)^2 - 3/2]$
$= 2[(x + 5/4)^2 - (5/4)^2 - 3/2]$
$= 2[(x + 5/4)^2 - 25/16 - 24/16]$
$= 2[(x + 5/4)^2 - 49/16]$.

Whereas in Problem 6C the working continued by applying the ‘difference of two squares’, here we will find the roots.

Thus the roots defined by

$2x^2 + 5x - 3 = 0$

can equivalently be found from

$2[(x + 5/4)^2 - 49/16] = 0$

Hence

$(x + 5/4)^2 - 49/16 = 0$
$(x + 5/4)^2 = 49/16$
$x + 5/4 = \pm 7/4$
$x = (-5 \pm 7)/4$

Thus, the roots are $x = -3$ and $x = +½$. From section 7.b) this implies that a factorisation of the original expression must be

$(+2)(x - (-3))(x - (+½))$
$= 2(x + 3)(x - ½)$.
$= (x + 3)(2x - 1)$, □

as found before.

Although this may be interesting and good practice — and a way to independently check the values of the roots — it clearly involves more work than direct application of the quadratic formula (as in Problem 8C).

9. Factorising using a graph to find the roots

Problem 9A: Monic quadratic with few factors

Factorise $x^2 - 8x + 7$.

Let’s start by plotting a few points for

$y = x^2 - 8x + 7$,

and sketching in a curve to fit these.

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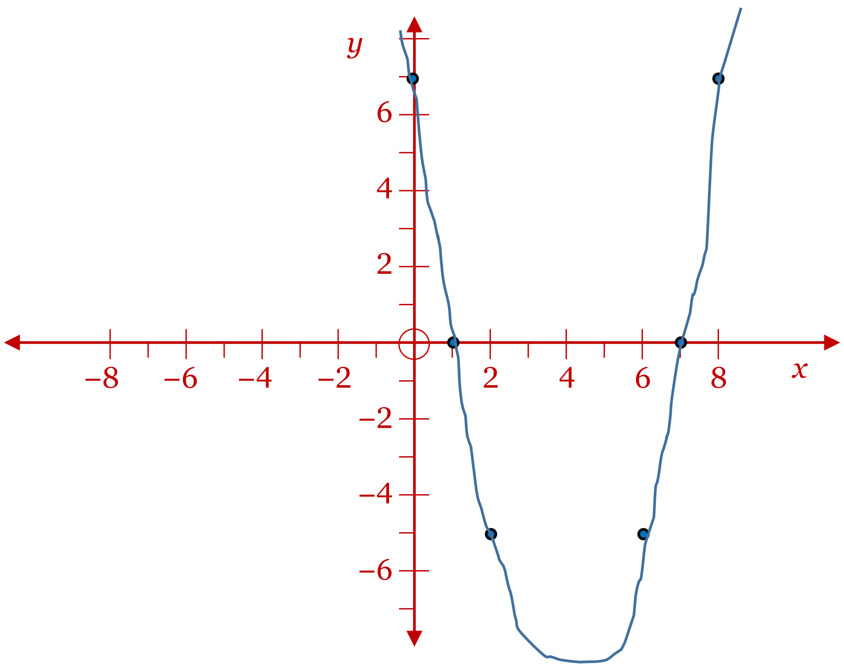


The curve crosses the x-axis at $x = +1$ and $x = +7$ — these are the x-intercepts.
Thus we know that the two roots are $x = +1$ and $x = +7$. From section 7.b) this implies that the factorisation of the original expression must be

$$(+1)(x-(+1))(x-(+7))$$
$$= (x-1)(x-7)\,.$$

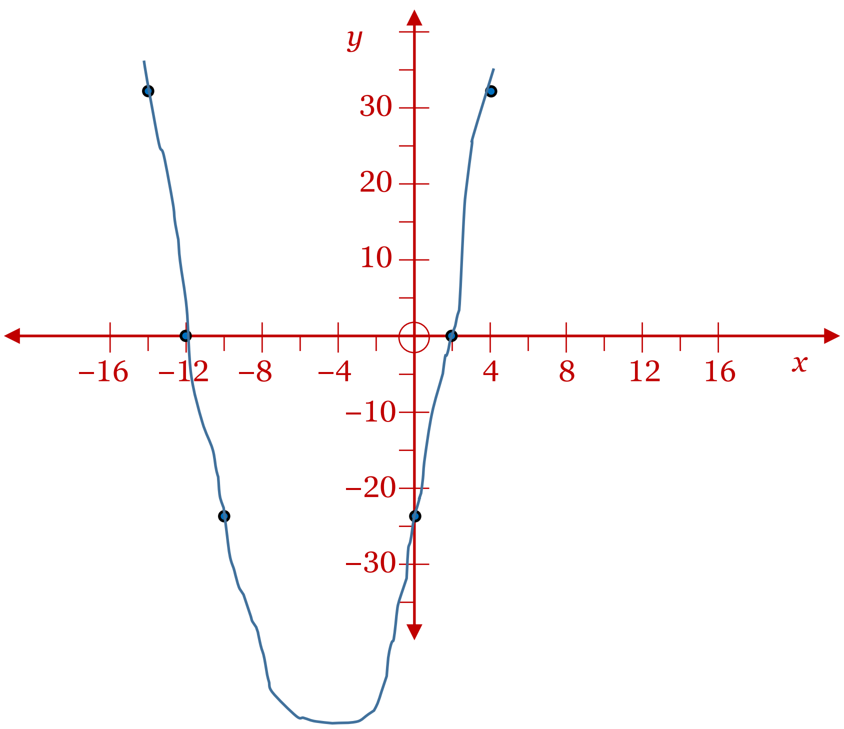
□

Problem 9B: Monic quadratic with many factors to consider

Factorise $x^2 + 10x - 24$.

Let's start by plotting a few points for

$$y = x^2 + 10x - 24\,,$$



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and sketching in a curve to fit these.

The curve crosses the x-axis at $x = -12$ and $x = +2$ — these are the x-intercepts.

This confirms that the roots are $x = -12$ and $x = +2$. From section 7.b) this implies that the factorisation of the original expression must be

$$(+1)(x - (-12))(x - (+2))$$
$$= (x + 12)(x - 2) \,.$$ □

Problem 9C: Non-monic quadratic with few factors to consider

Factorise $2x^2 + 5x - 3$.

This can be plotted and sketched in the same manner as for Problems 9A and 9B.

The curve crosses the x-axis at $x = -3$ and $x = +½$ — these are the x-intercepts.

This confirms that the roots are $x = -3$ and $x = +½$. From section 7.b) this implies that a factorisation of the original expression must be

$$(+2)(x - (-3))(x - (+\tfrac{1}{2}))$$
$$= 2(x + 3)(x - \tfrac{1}{2}) \,.$$

This is a mathematically satisfactory answer. Nonetheless, we can multiply the prefactor of +2 into the terms in the second set of parentheses to avoid including fractions

$$= (x + 3)(2x - 1) \,,$$ □

as found before.

Problem 9D: Non-monic quadratic with many factors to consider

Factorise $56x^2 - 158x + 45$.

This can be plotted and sketched in a similar manner as for Problems 9A and 9B.

We know from the other methods that the roots are $x = +9/28$ and $x = +2½$, so these are the x-intercepts — where the curve crosses the x-axis.

However, it is likely that only the right-hand intercept will be exactly found by plotting. **It is impractical to accurately determine the left-hand intercept graphically for this problem.** We might indeed recognise that the left-hand intercept falls somewhere between $x = +¼$ and $x = +½$, in the vicinity of $x = +0.3$ and $x = +⅓$. But neither of the latter two values are exactly right. If we plot manually to a precision of three decimal places in x, then we will identify that the intercept is very close to $x = 0.321$, but that is still not exactly right. Even using a computer software package to produce a magnified graph that reveals the intercept to be approximately $x = +0.321429$ would not provide an exact result — nor would it provide a strong clue that the required result is $x = +\frac{9}{28}$.

It turns out that the exact representation of the left-hand intercept in decimal form would be $x = 0.32\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$, in which the block of six dotted numbers repeats *ad infinitum*, so $x \approx 0.32142857142857142857142857142857$.

Practically speaking, it will not be possible to accurately solve this problem by graphing *alone*.

Refer to Problem 10D (pp. 26ff.) for further elucidation.

10. Factorising using trial-and-error to find roots

This is practically equivalent to plotting points on a graph, without actually drawing the graph.

Problem 10A: Monic quadratic with few factors

Factorise $x^2 - 8x + 7$.

Write this as a function, defined as

$$f(\,x\,) = x^2 - 8x + 7 \,.$$

Substitute values of x to try to obtain $f(\,x\,) = 0$.

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Try a few arbitrary (convenient) values:

$f(-10) = (-10)^2 - 8(-10) + 7 = +187$

$f(0) = (0)^2 - 8(0) + 7 = +7$

$f(+10) = (+10)^2 - 8(+10) + 7 = +27$

These few calculations suggest that if roots exist, they are more likely to be found in the vicinity of $x = 0$, perhaps in the range of $0 < x < +10$. Try a few more values:

$f(-10) = (-10)^2 - 8(-10) + 7 = +187$

$f(0) = (0)^2 - 8(0) + 7 = +7$

$f(+2) = (+2)^2 - 8(+2) + 7 = -5$

$f(+5) = (+4)^2 - 8(+4) + 7 = -9$

$f(+7) = (+7)^2 - 8(+7) + 7 = 0$

$f(+10) = (+10)^2 - 8(+10) + 7 = +27$

The values worked out previously are duplicated above to emphasise the characteristics by which $f(x)$ changes as x is changed.

By chance we have been fortunate to hit upon $f(+7) = 0$, indicating that $x = +7$ is one of the roots we sought.

If the values of $f(x)$ on either side of $f(+7)$ had the same sign, then we would have to conclude that an extremum exists at $x = +7$, meaning that there is only one value for the root of this equation. In fact, we see that $f(+5)$ and $f(+10)$ do not have the same sign, so there is a crossing of the x-axis at $x = +7$, and thus there must be another crossing elsewhere.

We observe that $f(0)$ gave a positive result, whereas $f(+2)$ gave a negative result. We know from this that the other root must satisfy $0 < x < +2$. The root could be rational or it could be irrational; it could be an integer or a non-integer — we do not yet know. However, if we don't have any other information, then we should follow two principles: (i) choose convenient values; (ii) try to halve the interval. The choice $x = +1$ is midway between 0 and +2, and it is also easy to calculate $f(+1)$, so this is an obvious choice for the next trial.

$f(+1) = (+1)^2 - 8(+1) + 7 = 0$

Again by good fortune, this turns out to be a root.

As a matter of presentation, the above could optionally be neatly organised into a table of values — just like when preparing to plot points of the parabola on a graph.

x	−10	0	+1	+2	+5	+7	+10
$f(x)$	+187	+7	0	−5	−9	0	+27

Whichever way they're presented, it certainly helps to gain insight if the x values are written in a consistent sequence.

Now we know that the two roots are $x = +1$ and $x = +7$. From section 7.b) this implies that the factorisation of the original expression must be

$(+1)(x - (+1))(x - (+7))$

$= (x - 1)(x - 7)$. □

Problem 10B: Monic quadratic with many factors to consider

Factorise $x^2 + 10x - 24$.

Write this as a function, defined as

$f(x) = x^2 + 10x - 24$.

Substitute values of x to try to obtain $f(x) = 0$.

Try a few arbitrary (convenient) values:

$f(-10) = (-10)^2 + 10(-10) - 24 = -24$

$f(0) = (0)^2 + 10(0) - 24 = -24$

$f(+10) = (+10)^2 + 10(+10) - 24 = +176$

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There is an interesting symmetry that we have accidentally uncovered with f(−10) = f(0), indicating that the turning point of this parabola lies on $x = -5$.

Looking at the signs, there must be one root between $x = 0$ and $x = +10$, because $-24 < 0 < +176$. Perhaps it will be a bit closer to $x = 0$, so let's try $x = +4$. Knowing that the symmetry axis is $x = -5$, we will also therefore be interested in exploring for the second root in the vicinity of $x = -5 - (+4 - (-5)) = -14$.

Both of these x values, +4 and −14, are located 9 units away from the axis of symmetry at $x = -5$.

$$f(-14) = (-14)^2 + 10(-14) - 24 = +32$$
$$f(-10) = (-10)^2 + 10(-10) - 24 = -24$$
$$f(0) = (0)^2 - 8(0) - 24 = -24$$
$$f(+4) = (+4)^2 + 10(+4) - 24 = +32$$
$$f(+10) = (+10)^2 + 10(+10) - 24 = +176$$

The good news is that the symmetry we noticed before is confirmed here. However, we have not yet struck a root. Now we are confident that one root must satisfy $-14 < x < -10$, while the other root must satisfy $0 < x < +4$. It has been mentioned that the most efficient strategy, in the absence of any other information, is to take a value in the middle; *i.e.* now we should try $x = -12$ and $x = +2$.

$$f(-14) = (-14)^2 + 10(-14) - 24 = +32$$
$$f(-12) = (-12)^2 + 10(-12) - 24 = 0$$
$$f(-10) = (-10)^2 + 10(-10) - 24 = -24$$
$$f(0) = (0)^2 + 10(0) - 24 = -24$$
$$f(+2) = (+2)^2 + 10(+2) - 24 = 0$$
$$f(+4) = (+4)^2 + 10(+4) - 24 = +32$$
$$f(+10) = (+10)^2 + 10(+10) - 24 = +176$$

Again, as a matter of presentation, the above could optionally be neatly organised into a table of values.

x	−14	−12	−10	0	+2	+4	+10
f(x)	+32	0	−24	−24	0	+32	+176

This confirms that the roots are $x = -12$ and $x = +2$. From section 7.b) this implies that the factorisation of the original expression must be

$$(+1)(x - (-12))(x - (+2))$$
$$= (x + 12)(x - 2).$$

□

Problem 10C: Non-monic quadratic with few factors to consider

Factorise $2x^2 + 5x - 3$.

Write this as a function, defined as

$$f(x) = 2x^2 + 5x - 3.$$

Substitute values of x to try to obtain $f(x) = 0$.

Try a few arbitrary (convenient) values:

$$f(-10) = 2(-10)^2 + 5(-10) - 3 = +147$$
$$f(0) = 2(0)^2 + 5(0) - 3 = -3$$
$$f(+10) = 2(+10)^2 + 5(+10) - 3 = +247$$

From this we might surmise that there is a minimum in $f(x)$ at around about $x = -1$, and roots may lie symmetrically around this at about $x = -3$ and $x = +1$. Trying these we find:

$$f(-10) = 2(-10)^2 + 5(-10) - 3 = +147$$
$$f(-3) = 2(-3)^2 + 5(-3) - 3 = 0$$
$$f(0) = 2(0)^2 + 5(0) - 3 = -3$$
$$f(+1) = 2(+1)^2 + 5(+1) - 3 = +4$$
$$f(+10) = 2(+10)^2 + 5(+10) - 3 = +247$$

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We have established that one root exists at $x = -3$.

The value of the other root is not yet known, but we can see that it must satisfy $0 < x < +1$, because $-3 < 0 < +4$. Therefore the second root cannot be an integer. It could have any non-integer value, but a sensible next choice is to halve the interval and try $x = +½$.

$$f(+½) = 2(+½)^2 + 5(+½) - 3 = 0$$

This above results can optionally be written in a table of values.

x	−10	−3	0	+½	+1	+10
f(x)	+147	0	−3	0	+4	+247

This confirms that the roots are $x = -3$ and $x = +½$. From section 7.b) this implies that a factorisation of the original expression must be

$$(+2)(x - (-3))(x - (+½))$$
$$= 2(x + 3)(x - ½) .$$

This is a mathematically satisfactory answer. Nonetheless, we can multiply the prefactor of 2 into the terms in the second set of parentheses to avoid including fractions

$$= (x + 3)(2x - 1) ,$$ □

as found before.

Problem 10D: Non-monic quadratic with many factors to consider

Factorise $56x^2 - 158x + 45$.

Write this as a function, defined as

$$f(x) = 56x^2 - 158x + 45 .$$

Substitute values of x to try to obtain $f(x) = 0$.

Try a few arbitrary (convenient) values:

$$f(-10) = 56(-10)^2 - 158(-10) + 45 = +7225$$
$$f(0) = 56(0)^2 - 158(0) + 45 = +45$$
$$f(+10) = 56(+10)^2 - 158(+10) + 45 = +4065$$

It is not clear from this whether there the quadratic actually has any roots, but if they exist then they should be located where $f(x)$ is closest to zero, which seems to be in the neighbourhood of $x = +1$. Let's try $x = +1$ and $x = +2$.

$$f(-10) = 56(-10)^2 - 158(-10) + 45 = +7225$$
$$f(0) = 56(0)^2 - 158(0) + 45 = +45$$
$$f(+1) = 56(+1)^2 - 158(+1) + 45 = -57$$
$$f(+2) = 56(+2)^2 - 158(+2) + 45 = -47$$
$$f(+10) = 56(+10)^2 - 158(+10) + 45 = +4065$$

Based on the signs of $f(x)$ we now have confirmation that there are two roots. One root is between 0 and +1; the other root is clearly between +2 and +10, however by consideration of symmetry we can expect it will lie between +2 and +3. Hence, let's try $x = +½$, $x = +2½$, and $x = +3$.

$$f(-10) = 56(-10)^2 - 158(-10) + 45 = +7225$$
$$f(0) = 56(0)^2 - 158(0) + 45 = +45$$
$$f(+½) = 56(+½)^2 - 158(+½) + 45 = -20$$
$$f(+1) = 56(+1)^2 - 158(+1) + 45 = -57$$
$$f(+2) = 56(+2)^2 - 158(+2) + 45 = -47$$
$$f(+2½) = 56(+2½)^2 - 158(+2½) + 45 = 0$$
$$f(+3) = 56(+3)^2 - 158(+3) + 45 = +75$$
$$f(+10) = 56(+10)^2 - 158(+10) + 45 = +4065$$

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One root is thus located at $x = +2\frac{1}{2}$. The other must satisfy $0 < x < +\frac{1}{2}$. Perhaps $x = +\frac{1}{4}$?

$f(-10) = 56(-10)^2 - 158(-10) + 45 = +7225$

$f(0) = 56(0)^2 - 158(0) + 45 = +45$

$f(+\frac{1}{4}) = 56(+\frac{1}{4})^2 - 158(+\frac{1}{4}) + 45 = +9$

$f(+\frac{1}{2}) = 56(+\frac{1}{2})^2 - 158(+\frac{1}{2}) + 45 = -20$

$f(+1) = 56(+1)^2 - 158(+1) + 45 = -57$

$f(+2) = 56(+2)^2 - 158(+2) + 45 = -47$

$f(+2\frac{1}{2}) = 56(+2\frac{1}{2})^2 - 158(+2\frac{1}{2}) + 45 = 0$

$f(+3) = 56(+3)^2 - 158(+3) + 45 = +75$

$f(+10) = 56(+10)^2 - 158(+10) + 45 = +4065$

The above results can optionally be written in a table of values.

x	−10	0	+¼	+½	+1	+2	+2½	+3	+10
f(x)	+7225	+45	+9	−20	−57	−47	0	+75	+4065

So the second root must satisfy $+\frac{1}{4} < x < +\frac{1}{2}$.

This second root is evidently going to be difficult to determine. There are a few options we could try:

(i) continue trying other values of x — either fractions (one third, two-fifths, three-eighths, *etc.*), or decimals (*e.g.* 0.30, 0.35, 0.40, 0.45); **or**

(ii) use polynomial long-division to divide the original expression by a term obtained from the known root, namely $(x - (+2\frac{1}{2}))$, that is, $(x - +2\frac{1}{2})$; **or**

(iii) work backwards from a partially complete factorisation (analogous to the PSF/ACE method).

From the discussion of Problem 9D (p. 23), option i is not suitable for this problem. (Option i could indeed be appropriate for other problems, although generally it's not possible to know in advance how suitable it will be!) Option ii will work, but is beyond the scope of introductory high school courses on algebraic factorisation. Let's choose option iii.

The roots are $x = +2\frac{1}{2}$ and $x = Q$ (Q is yet to be determined). From section 7.b) this implies that a factorisation of the original expression must be

$(+56)(x - (+2\frac{1}{2}))(x - Q)$

$= 56(x - 2\frac{1}{2})(x - Q)$.

Now we have a much easier problem: the product $56 \times -2\frac{1}{2} \times -Q = 140Q$ must be equal to the constant in the original polynomial (+45). Hence, $Q = +45/140 = +9/28$. The second root is $x = +9/28$.

Note: 9/28 can be represented as a recurring decimal number, *approximately* equal to 0.32142857142857142857142857 . Thus it would be **practically impossible** to find *exactly* by trial-and-error choice of decimal values. It is also not a commonly encountered fraction, so it would be very time-consuming to try to find this root as a fraction by trial-and-error.

We now have a complete factorisation:

$= 56(x - 2\frac{1}{2})(x - 9/28)$.

This is a mathematically satisfactory answer. Nonetheless, we can avoid including fractions by multiplying through as follows

$= 2(x - 2\frac{1}{2})(28x - 9)$.

$= (2x - 5)(28x - 9)$, □

as found before.

11. Comparison of factorisation methods

Method	Advantage(s)	Disadvantage(s)
Preparatory measures		
1. Common factor (pp. 2ff.)	Easy. Often helpful.	Typically not enough on its own.
Trialling the factors		
3. Factorising by trial-and-error ("guess-and-check") (pp. 4ff.)	Easy for "**simple**" quadratics*.	May lose track of factors tried and those yet to be tried. Laborious for "**complicated**" quadratics†.
4. Factorising by trial-and-error, using a cross layout:	Easy for simple quadratics.	Laborious for complicated quadratics.
• Large schematic presentation (pp. 7f.)	Visually depicts relationships.	Takes time and space to draw up template(s). May lose track of factors tried and those yet to be tried.
• Compact tabular presentation (pp. 8ff.)	Supports systematically working through options.	
5. Factorising using the PSF method (ACE method):	Once the pairs of factors have been generated, it is obvious which to choose.	It's more difficult to generate the pairs of factors: may need to consider factors of large numbers for complicated quadratics. May be 'overkill' for simple quadratics.
• Conventional implementation of the PSF method (ACE method) (pp. 10ff.)	Fewer steps in algorithm.	No need for pairwise factorisation.
• Alternative implementation of the PSF method (ACE method) (pp. 13ff.)	No need for pairwise factorisation.	More steps in algorithm.
6. Factorising by 'completing the square' then using the 'difference of two squares' (pp. 15ff.)	Impress your family and friends?	'Overkill' for simple quadratics (requires more working). For complicated quadratics requires a lot of cumbersome working with fractions and large numbers (multiplication, simplification and square root), with multiple levels of parentheses/brackets — increasing the risk of a mistake.

* That is, when there are relatively *few* pairs of factors (and combinations thereof, in the case of non-monic quadratics) to consider, and roots are integers or simple fractions.

† That is, when there are relatively *many* pairs of factors (and combinations thereof, in the case of non-monic quadratics) to consider, and roots may include complicated fractions.

Method	Advantage(s)	Disadvantage(s)
Identifying the roots		
8. Factorising using the 'quadratic formula' to find roots:	Guaranteed to work for *any* quadratic equation.	'Overkill' for simple quadratics (requires more working). May involve working with large numbers for complicated quadratics.
• Application of the quadratic formula (pp. 19ff.)	Requires one useful formula that is easy to remember (if you apply it regularly).	
• 'Completing the square' to obtain the roots (p. 21)	A backup plan if you forget the quadratic formula?	Much more work than just using the quadratic formula directly.
9. Factorising using a graph to find the roots (pp. 21ff.)	Provides great insight: there is a visual guide to the location of roots. A mistake in plotting one point will be obvious as a deviation from the expected shape of a parabola. After roots are found, working is quite easy.	May **fail** (if used on its own) for complicated quadratics.
10. Factorising using trial-and-error to find roots (pp. 23ff.)	Provides good insight when conducted systematically.	May **fail** (if used on its own) for complicated quadratics.

12. Checking your answers

a) Inspect your workings

The most common strategy is to inspect one's workings.

The advantages of this are that: it is a good technique for identifying the fundamental cause of any error that crept in; it avoids the feeling of 'wasting one's effort' by valuing the work already done; and if an error is identified, it is sometimes easy to make a few amendments to the workings and obtain the correct answer without starting all over again.

The disadvantages are that: it is often actually slower than other methods of checking; and mistakes can easily be overlooked (especially if the checking is done by the same person who made the original mistake!).

b) Try again by independently working through a different method

Several methods were presented herein. If the same answer is obtained independently by more than one method, then greater confidence can be had that the answer is indeed correct.

The main advantage of this approach is that it provides much more assurance of correctness, in the event of matching answers, than merely by inspecting one's workings for a single method.

The disadvantages of this approach are that: it can use up a substantial amount of time unnecessarily if there were not actually any mistakes in the original workings; if two methods yield different answers, it is not clear which of the two is correct; and the fundamental source of any error is not identified.

c) Expand out the factorised form

An obvious check is to ensure that the purported factorised form really does recover the original expression when it is expanded *.

The advantages of this approach are that: it is typically quite quick; and it provides much more assurance of correctness, in the event of matching expressions, than merely by inspecting one's workings for a single method.

The disadvantages of this approach are that: the fundamental source of any error is not identified; and no alternative answer is suggested.

d) Check equality of expressions for specific values of *x*

Substitute values of x into both the original (expanded) polynomial and the purported factorisation of it.

The number of values to test must exceed the maximum possible number of roots to be deemed a 'comprehensive' check. Thus, for a quadratic expression three distinct values of x need to be chosen for testing.

If fewer values are tested, then it is possible purely by chance to be able to equate both forms of the expression. The likelihood of this 'fluky' conclusion is reduced as more values are tested.

We have freedom to choose any values, so we might as well choose convenient ones. For example: $\{-1, 0, +1\}$; or $\{0, +1, +2\}$; or {first purported root, second purported root, other value}.

The advantages of this approach are that: it is typically quite quick; and it provides much more assurance of correctness, in the event of matching expressions, than merely by inspecting one's workings for a single method.

The disadvantages of this approach are that: the fundamental source of any error is not identified; and no alternative answer is suggested.

Example A

Verify that $x^2 - 8x + 7 = (x-1)(x-7)$.

For $x = -1$:

$\text{LHS} = (-1)^2 - 8(-1) + 7 = +1 + 8 + 7 = +16$

$\text{RHS} = ((-1)-1)((-1)-7) = (-2)(-8) = +16 = \text{LHS}.$

"LHS" means the expression on the left-hand side of the original equation, and "RHS" means that on the right-hand side.

For $x = 0$:

$\text{LHS} = (0)^2 - 8(0) + 7 = 0 + 0 + 7 = +7$

$\text{RHS} = ((0)-1)((0)-7) = (-1)(-7) = +7 = \text{LHS}.$

For $x = +1$:

$\text{LHS} = (+1)^2 - 8(+1) + 7 = +1 - 8 + 7 = 0$

$\text{RHS} = ((+1)-1)((+1)-7) = (0)(-6) = 0 = \text{LHS}.$

As we have proved that LHS = RHS for three different values of x, the equality must be true for any values of x — *i.e.* we have proved that the equation is true in general. □

Example B

Verify that $x^2 + 10x - 24 = (x+12)(x-2)$.

For $x = 0$:

$\text{LHS} = (0)^2 + 10(0) - 24 = 0 + 0 - 24 = -24$

$\text{RHS} = ((0)+12)((0)-2) = (12)(-2) = -24 = \text{LHS}.$

* This approach is emphasised in the mathcentre's workbook http://www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-factorisingquadratics.pdf

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For $x = +1$:

$\text{LHS} = (+1)^2 + 10(+1) - 24 = +1 + 10 - 24 = -13$

$\text{RHS} = ((+1) + 12)((+1) - 2) = (13)(-1) = -13 = \text{LHS}.$

For $x = +2$:

$\text{LHS} = (+2)^2 + 10(+2) - 24 = +4 + 20 - 24 = 0$

$\text{RHS} = ((+2) + 12)((+2) - 2) = (14)(0) = 0 = \text{LHS}.$

As LHS = RHS for three different values of x, equality for all circumstances is verified. □

Example C

Verify that $2x^2 + 5x - 3 = (x + 3)(2x - 1)$.

For $x = -3$:

$\text{LHS} = 2(-3)^2 + 5(-3) - 3 = 0 = ((-3) + 3)(2(-3) - 1) = \text{RHS}$

For $x = +\tfrac{1}{2}$:

$\text{LHS} = 2(+\tfrac{1}{2})^2 + 5(+\tfrac{1}{2}) - 3 = 0 = ((+\tfrac{1}{2}) + 3)(2(+\tfrac{1}{2}) - 1) = \text{RHS}$

For $x = 0$:

$\text{LHS} = 2(0)^2 + 5(0) - 3 = -3 = ((0) + 3)(2(0) - 1) = \text{RHS}$

As LHS = RHS for three different values of x, equality for all circumstances is verified. □

Example D

Verify that $56x^2 - 158x + 45 = (28x - 9)(2x - 5)$.

For $x = 0$:

$\text{LHS} = 56(0)^2 - 158(0) + 45 = +45 = (28(0) - 9)(2(0) - 5) = \text{RHS}$

For $x = +1$:

$\text{LHS} = 56(+1)^2 - 158(+1) + 45 = -57 = (28(+1) - 9)(2(+1) - 5) = \text{RHS}$

For $x = +7$:

$\text{LHS} = 56(+7)^2 - 158(+7) + 45 = +1683 = (28(+7) - 9)(2(+7) - 5) = \text{RHS}$

As LHS = RHS for three arbitrary values of x, equality for all circumstances is verified. □

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Australian Business Number (ABN):
64 011 074 054
https://tutoring.div1.com.au
P.O. Box 1353, Meadowbank, N.S.W. 2114, AUSTRALIA