Perils of Squaring Equations

Did your teacher ever tell you about the perils of squaring equations?
Squaring an equation typically (almost always) introduces *spurious solutions*. This means that besides the correct solution, an incorrect solution will also be produced.

1. General concept

a) A spurious negative result

What values of x are consistent with $x = 3$?

Clearly here we already know the (unique) answer is $x = 3$. But suppose we decided to square both sides of the equation.

$$(x)^2 = (3)^2$$

This is mathematically consistent with the original statement, but it is not as strict (as we shall soon see). Simplifying:

$$x^2 = 9$$

And then the key step is to recognise that this equation obtained by squaring both sides allows both −3 and +3 as solutions, which is evident when we (correctly!) apply the square root operation on both sides of this equation.

$$\sqrt{x^2} = \pm\sqrt{9}$$
$$x = \pm 3$$

But only one of these apparent solutions satisfies the original statement (which has priority); the other apparent solution is incorrect, so we can call it a 'spurious solution'.
To mathematically explain what went wrong, we need to reassert a constraint that was implicitly contained in the original expression (but not implicit in the squared equation):

$x = 3$

$(x)^2 = (3)^2$, subject to the condition that $x \geq 0$

$x^2 = 9$, subject to the condition that $x \geq 0$

$\sqrt{x^2} = \pm\sqrt{9}$, subject to the condition that $x \geq 0$

$x = \pm 3$, subject to the condition that $x \geq 0$

$x = 3$

b) A spurious positive result

It is just as possible to obtain a spurious positive-valued result.

What values of x are consistent with $x = -3$?

$x = -3$

$(x)^2 = (-3)^2$, subject to the condition that $x \leq 0$

$x^2 = 9$, subject to the condition that $x \leq 0$

$\sqrt{x^2} = \pm\sqrt{9}$, subject to the condition that $x \leq 0$

$x = \pm 3$, subject to the condition that $x \leq 0$

$x = -3$

If we had ignored the implicit constraint, then we would have accepted the spurious result $x = +3$.

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2. Solving a very simple algebraic equation

Solve $x + 2 = 3$.

It should be clear that there is one unique solution to the above equation:

$$x + 2 - 2 = 3 - 2$$

$$x = +1$$

But, as foreshadowed in the previous section, squaring the equation will produce additional spurious results — if the implicit constraint is ignored.

a) Squaring the equation as given

Let us square the equation

$(x + 2)^2 = 3^2$, subject to the condition that $x + 2 \geq 0$

Notice that this is still consistent with the correct solution presented above.

Expanding the left hand side ("**LHS**"):

$x^2 + 4x + 4 = 9$, subject to the condition that $x + 2 \geq 0$

Rearranging and factorising:

$x^2 + 4x - 5 = 0$, subject to the condition that $x + 2 \geq 0$

$(x + 5)(x - 1) = 0$, subject to the condition that $x + 2 \geq 0$

$x = -5, x = +1$, subject to the condition that $x \geq -2$

While $x = -5$ is a valid solution of the squared equation when the implied constraint is ignored, it clearly does not solve the original equation — hence it represents a spurious solution.

b) Manipulating the equation and then squaring

What happens if we add or subtract integers on both sides of the original equation before squaring? Let's look at a few permutations.

	Add to each side	New equation	Squared form	Factorised	Solutions	Implied constraint
	+4	$x+6=7$	$(x+6)^2=49$	$(x+13)(x-1)=0$	$x=-13, x=+1$	$x+6>0$
	+3	$x+5=6$	$(x+5)^2=36$	$(x+11)(x-1)=0$	$x=-11, x=+1$	$x+5>0$
	+2	$x+4=5$	$(x+4)^2=25$	$(x+9)(x-1)=0$	$x=-9, x=+1$	$x+4>0$
	+1	$x+3=4$	$(x+3)^2=16$	$(x+7)(x-1)=0$	$x=-7, x=+1$	$x+3>0$
	0	$x+2=3$	$(x+2)^2=9$	$(x+5)(x-1)=0$	$x=-5, x=+1$	$x+2>0$
	−1	$x+1=2$	$(x+1)^2=4$	$(x+3)(x-1)=0$	$x=-3, x=+1$	$x+1>0$
	−2	$x=1$	$x^2=1$	$(x+1)(x-1)=0$	$x=-1, x=+1$	$x>0$
◇	−3	$x-1=0$	$(x-1)^2=0$	$(x-1)(x-1)=0$	$x=+1$	$x-1=0$
	−4	$x-2=-1$	$(x-2)^2=1$	$(x-3)(x-1)=0$	$x=+1, x=+3$	$x-2<0$
	−5	$x-3=-2$	$(x-3)^2=4$	$(x-5)(x-1)=0$	$x=+1, x=+5$	$x-3<0$

For every case bar one (marked with a diamond), a spurious solution is generated.

The crucial difference in the exceptional case is that the right hand side ("**RHS**") was set equal to zero by rearranging the original equation, prior to squaring. Notably $0^2 = 0$, and $+0 = -0$, so ± 0 can just be written as "0".

This highlights one of the benefits of rearranging an equation so that one side is equal to zero.

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3. Solving a quadratic equation

a) General case, with integer roots

Solve $x^2 - 5x = -6$.

The most efficient solution process is as follows:

$x^2 - 5x + 6 = 0$

$(x - 2)(x - 3) = 0$

$x = +2, x = +3$

As the original equation was quadratic, we should not be surprised to find two valid solutions.

What happens if we decide to square both sides first?

$(x^2 - 5x)^2 = (-6)^2$

$x^4 - 10x^3 + 25x^2 = 36$

$x^4 - 10x^3 + 25x^2 - 36 = 0$

(Implicit constraints still exist, but can't be stated just by inspecting the original equation.)

Solving quartics is rather tiresome, and typically involves 'guessing' one of the roots as the first step. Fortunately here we already know two of the roots, so both $(x - 2)$ and $(x - 3)$ must be factors of the quartic (confirmed by observing that x=+2 and x=+3 both satisfy the equation on the last line above). Applying polynomial long division twice yields:

$(x - 2)(x^3 - 8x^2 + 9x + 18) = 0$

$(x - 2)(x - 3)(x^2 - 5x - 6) = 0$

Note that the remaining quadratic corresponds to $x^2 - 5x = +6$ (compare to the original equation!). Factorising this yields:

$(x - 2)(x - 3)(x + 1)(x - 6) = 0$

Thus two spurious solutions have been generated: x=−1 and x=+6. Whereas the correct solutions are specified by $+2\frac{1}{2} \pm \frac{1}{2}$, the spurious solutions are given by $+2\frac{1}{2} \pm 3\frac{1}{2}$.

b) Perfect square

Solve $x^2 + 6x = -9$.

The most efficient solution process is as follows:

$x^2 + 6x + 9 = 0$

$(x + 3)(x + 3) = 0$

$(x + 3)^2 = 0$

$x = -3$

What happens if we decide to square both sides first?

$(x^2 + 6x)^2 = (-9)^2$

$x^4 + 12x^3 + 36x^2 = 81$

$x^4 + 12x^3 + 36x^2 - 81 = 0$

Proceeding as above:

$(x + 3)(x^3 + 9x^2 + 9x - 27) = 0$

$(x + 3)(x + 3)(x^2 + 6x - 9) = 0$

Note that the remaining quadratic corresponds to $x^2 + 6x = +9$ (compare to the original equation!). Using the quadratic formula to factorise further:

$(x + 3)(x + 3)(x + 3 + 3\sqrt{2})(x + 3 - 3\sqrt{2}) = 0$

Thus, two spurious solutions have been generated. Interestingly, they are each offset by $\pm 3\sqrt{2}$ from the genuine solution.

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c) Degenerate case

Solve $x^2 + 2x = 3$.

The most efficient solution process is as follows:

$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3, x = +1$$

As the original equation was quadratic, we should not be surprised to find two valid solutions.

What happens if we decide to square both sides first?

$$(x^2 + 2x)^2 = (3)^2$$
$$x^4 + 4x^3 + 4x^2 = 9$$
$$x^4 + 4x^3 + 4x^2 - 9 = 0$$

Proceeding as before to factorise the quartic yields:

$$(x + 3)(x^3 + x^2 + x - 3) = 0$$
$$(x + 3)(x - 1)(x^2 + 2x + 3) = 0$$

Note the similarity of the remaining quadratic to the original quadratic in the first line of the recommended solution. Considering the discriminant of the so-called 'quadratic formula', whereas the original quadratic could be factorised, it is apparent that the quadratic remaining now cannot be factorised.

So, by good fortune no spurious solutions have been generated. This was because the quartic arising by inverting the sign on the RHS of the original equation, yielding $x^2 + 2x = -3$, has no real solution. This reflects the capricious nature of solutions generated after squaring both sides of an equation — we don't *always* have to remove spurious solutions.

4. Motivation — practical examples

The preceding examples illustrate the generation of spurious results upon squaring equations, but in those equations the motivation to square the equation was not high. In the following examples the motivation to square the original equation is stronger, even though it is likely to lead to erroneous results.

a) Semicircle

Describe a sketch of $y = +\sqrt{4^2 - x^2}$, and state whether y is a function of x.

The above equation describes the upper half of a circle, *i.e.* a semicircle. It is a 'function', because each value of x corresponds to one unique value of y.

However, it would be tempting for some people to square both sides to obtain

$$x^2 + y^2 = 4^2 ,$$

which is the more familiar form of equation for a circle of radius 4, centred on the origin. Recognising this as the formula for a circle is likely to result in an incorrect sketch. Furthermore, the modified equation (for a circle) doesn't describe a function, as it corresponds to

$$y = \pm\sqrt{4^2 - x^2} .$$

To retain the original sense after squaring, the implicit constraint should be appended:

$$x^2 + y^2 = 4^2 , \qquad \text{subject to the condition that } y \geq 0$$

Unlike the previous examples which related to discrete numbers, here an entire range of results is involved. In the previous examples it would be possible to validate each candidate solution in turn by substitution into the respective original equation; that option is not readily apparent here — although 'test points' could still be checked for consistency.

b) Trigonometric identities, Part I

Solve $5\sin(\theta) = \sqrt{5^2 - 5^2\cos^2(\theta)}$ for θ.

First it is sensible to factorise and simplify:

$$5\sin(\theta) = \sqrt{5^2\,[1 - \cos^2(\theta)]}$$
$$5\sin(\theta) = 5\sqrt{1 - \cos^2(\theta)}$$
$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)}\ .$$

Let's review the feasible values of $\sin(\theta)$, $\cos(\theta)$ and θ. Under the square root we require a non-negative argument. $\cos^2(\theta)$ takes values from 0 to 1, and for all of these values the square root is defined, so this does not impose any limitation upon θ. Overall the RHS can only be positive or zero, so $\sin(\theta)$ cannot be negative, and thus θ must be in first or second quadrant.

There is a trigonometric identify that is valid for any angle:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \qquad \square$$

which is equivalent to

$$\sin(\alpha) = \pm\sqrt{1 - \cos^2(\alpha)}\ . \qquad \diamond$$

The original expression is equivalent to the identity (which would be valid for every value of θ), except that it imposes the additional constraint mentioned above. Therefore the correct solution is: $0 \le \theta + 2n\pi \le \pi$, in which n is any integer (positive, negative or zero).

If we had simply squared the original expression, we should have then also ensured the constraint remained in place, as in

$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$$
$$\sin^2(\theta) = 1 - \cos^2(\theta)\ \text{, subject to the condition that } \sin(\theta) \ge 0\ .$$

c) Trigonometric identities, Part II

Solve $\sqrt{1 - \cos^2(\theta)}\ /\ \cos(\theta) = -1$ for θ.

Analysis of the signs involved reveals that on the left side the numerator is positive, so the denominator must be negative in order to match the RHS. Thus, the implicit constraint is

$$\cos(\theta) < 0\ .$$

This means that the solutions can only be in the second and third quadrants (Q2 and Q3).

Solution method 1 (without squaring, in two steps)

Using the *positively* signed equality indicated in section 4.b) (equation marked '⋄'),

$$+\sqrt{1 - \cos^2(\theta)} = +\sin(\theta)\ , \qquad \text{with } \theta \text{ in Q1 or Q2,}$$

and

$$\sqrt{1 - \cos^2(\theta)}\ /\ \cos(\theta) = -1\ , \qquad \text{with } \theta \text{ in Q2 or Q3,}$$

we obtain

$$\sin(\theta)\ /\ \cos(\theta) = -1\ , \qquad \text{with } \theta \text{ in Q2 only.}$$

Hence

$$\tan(\theta) = -1\ , \qquad \text{with } \theta \text{ in Q2 only.}$$

Thus $\theta = 135° = 3\pi/4$ is a valid solution, and therefore all values $\theta = 3\pi/4 + 2n\pi$ satisfy the original equation, where n is any integer (positive, negative or zero). However, this is not the complete solution!

Consider a simple rearrangement of the original equation:

$$-\sqrt{1 - \cos^2(\theta)}\ /\ \cos(\theta) = +1$$

Using the *negatively* signed equality indicated in section 4.b) (equation marked '⋄'),

$$-\sqrt{1 - \cos^2(\theta)} = +\sin(\theta)\ , \qquad \text{with } \theta \text{ in Q3 or Q4,}$$

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and

$-\sqrt{1-\cos^2(\theta)}\,/\,\cos(\theta) = +1$, with θ in Q2 or Q3,

we obtain

$+\sin(\theta)\,/\,\cos(\theta) = +1$, with θ in Q3 only.

Hence

$\tan(\theta) = +1$, with θ in Q3 only.

Thus $\theta = 225° = 5\pi/4$ is a valid solution, and therefore all values $\theta = 5\pi/4 + 2n\pi$ satisfy the original equation, where n is any integer (positive, negative or zero).

Consequently the complete final solution is $\theta = (1 \pm ¼)\pi + 2n\pi = (2n + 1 \pm ¼)\pi$.

Solution method 2 (without squaring, in one step)

The equation marked '◇' in section 4.b) can be rearranged slightly,

$\sqrt{1-\cos^2(\alpha)} = \pm\sin(\alpha)$.

So the problem to be solved here can now be written

$[\pm\sin(\theta)]\,/\,\cos(\theta) = -1$, with θ in Q2 or Q3

$\sin(\theta)\,/\,\cos(\theta) = \pm 1$, with θ in Q2 or Q3.

Hence

$\tan(\theta) = \pm 1$, with θ in Q2 or Q3.

Thus $\theta = 135° = 3\pi/4$ and $\theta = 225° = 5\pi/4$ are immediately found as valid solutions. Therefore the complete final solution is $\theta = (1 \pm ¼)\pi + 2n\pi = (2n + 1 \pm ¼)\pi$, as found before.

Solution method 3 (squaring both sides)

Square both sides

$[1 - \cos^2(\theta)]\,/\,\cos^2(\theta) = 1$, with θ in Q2 or Q3.

Therefore, using the identity (the equation marked '□' above) yields

$\sin^2(\theta)\,/\,\cos^2(\theta) = 1$, with θ in Q2 or Q3.

Hence

$[\sin(\theta)\,/\,\cos(\theta)]^2 = 1$, with θ in Q2 or Q3,

$[\tan(\theta)]^2 = \tan^2(\theta) = 1$, with θ in Q2 or Q3.

Using caution in then taking the square root of both sides

$\tan(\theta) = \pm 1$, with θ in Q2 or Q3.

If not for the implicit constraint, $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ would all be valid solutions. However, substitution confirms that $\theta = \pi/4$ and $\theta = 7\pi/4$ (and any equivalent angles differing by $2n\pi$) are erroneous solutions that do not satisfy the original equation.

By applying the implicit constraint we retain only $\theta = 3\pi/4$ and $\theta = 5\pi/4$ as valid solutions, and hence the final complete solution is $\theta = 3\pi/4 + 2n\pi$ and $\theta = 5\pi/4 + 2n\pi$, where n is any integer (positive, negative or zero). This can equivalently be written $\theta = (1 \pm ¼)\pi + 2n\pi = (2n + 1 \pm ¼)\pi$.

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